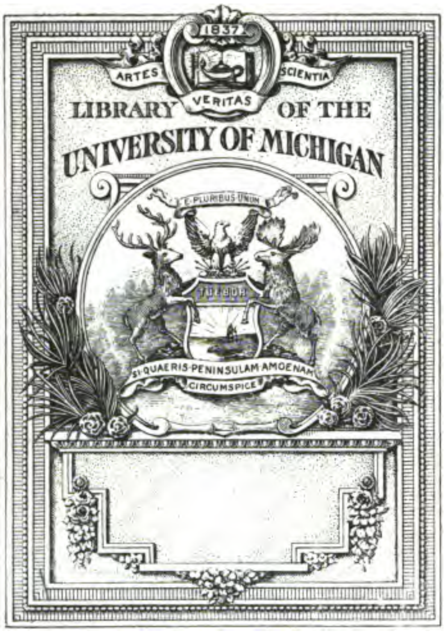

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MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. MILLER, B.A.,

MATHEMATICAL MASTER, HUDDERSFIELD COLLEGE.

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WRIGHT, Rev. R. H., M.A., Head Master of Ashford Grammar School.

CONTENTS.

CORRECTIONS and ADDITIONS to be made in Vols. I., II., III. of the *Reprint*.

VOL. I.

p. 33, Quest. 1387, line 5, before "show," insert "if A, B be *opposite* intersections of the tangents."

VOL II.

p. iv. (Contents), read Quest. (1484......pp. 39 & 85.)
p. vii., insert Quest. (1517......p. 79), instead of Quests. (1521, 1530) which are on p. 14 of Vol. III.
p. 3, line 25, for γ read r.
p. 5, line 4 from bottom, read $Bp < BP$.
p. 10, line 8, insert C.
p. 30, line 18, eq. 3, for $(2x-x)$ read $(2a-x)$.
p. 31, line 16, eq. 3, for m read n.
p. 31, last line, the *Radial* reduces to $q \cot\theta = \dfrac{2r(\cos^3\theta + \sin^3\theta) + a}{2r(\cos^3\theta + \sin^3\theta) - a}$.
p. 32, line 7, for θ read $(\frac{1}{4}\pi + \theta)$.
p. 32, line 10, for $\sec\dfrac{n}{1-n}$ read $\sec\dfrac{n\theta}{1-n}$.
p. 34, line 16, for *which......angle*, read *and the curve and its Radial are complementally inclined to the axes.*
p. 67, line 4, insert $\therefore$ SQ before $=$.
p. 73, line 2 from bottom, insert a^2 before sin A cos A.
p. 95, line 8 from bottom, insert (2) before $dz\,dy\,dx$.

VOL. III.

p. 33, line 3, for $-$ read $=$.
p. 51, line 3 from bottom, for *geographical* read *geometrical.*
p. 97, Quest. 1708, add to (2) the following

NOTE. When four normals to an ellipse, at the points whose eccentric angles are α, β, γ, δ, meet in a point, we have
$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0 = \sin(\beta+\delta) + \sin(\delta+\alpha) + \sin(\alpha+\beta)$
whence we obtain $\alpha+\beta+\gamma+\delta = n\pi$.

p. 107, line 20, read $(bC)^{-1} - (Bc)^{-1}$.
p. 110, Quest. 1702, line 1, before "prove," insert "If a, b, c be the sides, and A, B, C the angles of a triangle."
p. 111, line 1, the number of this Question should be 1668, instead of 1679.

MATHEMATICAL QUESTIONS

WITH THEIR

Solutions.

FROM

THE EDUCATIONAL TIMES.

1567. (Proposed by the Rev. ROBERT HARLEY, F.R.S.)—
If $[m]^r = m(m-1)\ldots.(m-r+1)$ as usual; prove that

$$[m]^r - (r+1)[m-1]^r + \frac{(r+1)r}{1.2}[m-2]^r - \frac{(r+1)r(r-1)}{1.2.3}[m-3]^r + \ldots. + (-)^{r+1}[m-r-1]^r = 0;$$

and show how to extend and generalize the theorem.

I. *Solution by* MR. S. BILLS.

Instead of the Question as proposed, we shall investigate the following more general theorem (A).

Let the series $1,\ (r+1),\ \dfrac{(r+1)r}{1.2},\ \dfrac{(r+1)r(r-1)}{1.2.3}$, &c. &c., be continued to $(r+2)$ terms, or until the last term becomes unity; then, if $a, b, c, d \ldots. n$ denote *any numbers whatever*, either *whole* or *fractional*, we shall have, universally,

$$\begin{aligned} &a.b.c\ldots\ldots n - (r+1)(a-1)(b-1)(c-1)\ldots\ldots(n-1) \\ &\quad + \frac{(r+1)r}{1.2}(a-2)(b-2)(c-2)\ldots\ldots(n-2) \\ &\quad - \frac{(r+1)r(r-1)}{1.2.3}(a-3)(b-3)(c-3)\ldots\ldots(n-3) \\ &\ldots\ldots\ldots \pm (a-r-1)(b-r-1)(c-r-1)\ldots\ldots(n-r-1) = 0 \ldots. (A). \end{aligned}$$

DEMONSTRATION.—We have, as is well known,

$$1-(r+1)+\frac{(r+1)\,r}{1\,.\,2}-\frac{(r+1)\,r\,(r-1)}{1\,.\,2\,.\,3}\ldots\ldots\pm 1=(1-1)^{r+1}=0\ldots(1);$$

$$1-r+\frac{r\,(r-1)}{1\,.\,2}-\frac{r\,(r-1)\,(r-2)}{1\,.\,2\,.\,3}\ldots\ldots\mp 1=(1-1)^{r}=0\quad\ldots.(2).$$

Now multiply (1) by a, and (2) by $(r+1)$, and add the results; then

$$a-(r+1)\,(a-1)+\frac{(r+1)\,r}{1\,.\,2}\,(a-2)\ldots\ldots\pm(a-r-1)=0\quad\ldots\ldots..(3).$$

Similarly $(a-1)-r\,(a-2)+\dfrac{r\,(r-1)}{1\,.\,2}\,(a-3)\ldots.\mp(a-r-1)=0\ldots(4).$

Multiply (3) by b, and (4) by $(r+1)$, and add the results; then

$$ab-(r+1)\,(a-1)\,(b-1)+\frac{(r+1)\,r}{1\,.\,2}\,(a-2)\,(b-2)\ldots\ldots$$
$$\pm(a-r-1)\,(b-r-1)=0\ldots\ldots\ldots.(5).$$

In like manner we obtain

$$(a-1)\,(b-1)-r\,(a-2)\,(b-2)+\frac{r\,(r-1)}{1\,.\,2}\,(a-3)\,(b-3)\ldots\ldots$$
$$\mp(a-r-1)\,(b-r-1)=0\ldots\ldots\ldots.(6).$$

By proceeding in the same manner with the equations (5), (6), &c., introducing the new letters c, d, &c., to n, we obtain, finally, the general theorem (A), of which the theorem in the Question is evidently a particular case.

As an example, in numbers, of the truth of the above theorem, let $r=4$, and suppose there to be four letters, viz., $a=8$, $b=7\tfrac{1}{2}$, $c=7$, and $d=6$; then, by substitution, the theorem becomes

$$2520-6825+6600-2700+420-15=0.$$

It is proper to observe that it is essential to the truth of the theorem that the *number* of letters a, b, c, &c. must not *exceed the value of* r.

II. *Solution by the* REV. J. BLISSARD.

The proposed Formula is capable of being variously extended and generalized.

Assume $u_n=[n]^r=n\,(n-1)\ldots.(n-r+1)$; then, r being a positive integer, $u_0, u_1, u_2, \ldots u_{r-1}$ all vanish, $u_r=1\,.\,2\ldots r$, $u_{r+1}=2\,.\,3\ldots(r+1)$, and so on. Hence, using Representative Notation,

$$u^0\epsilon^{ux}=u_0+u_1x+u_2\frac{x^2}{1\,.\,2}+\ldots.+u_r\frac{x^r}{1\,.\,2\ldots r}+\ldots.$$

$$=x^r+x^{r+1}+\frac{x^{r+2}}{1\,.\,2}+\ldots.=x^r\epsilon^x;$$

$$\therefore\ u^0\epsilon^{(u-1)\,x}=x^r.$$

Expanding and equating coefficients, $u^0(u-1)^n = 0$, unless $n=r$, in which case $u^0(u-1)^r = 1 \,.\, 2 \ldots r$. Hence, if fu be any function of u, and (which can always be done) be expanded in terms of ascending powers of $(u-1)$, and if in that expansion C_r be the coefficient of $(u-1)^r$,

$$u^0 fu = C_r u^0 (u-1)^r = 1 \,.\, 2 \ldots r \,.\, C_r \ldots\ldots\ldots\ldots\ldots\ldots (\text{I.})$$

This equation appears to be of considerable importance, since by varying the form of fu an indefinite number of general results may be obtained.

(1.) Let $fu = u^{m-n}(u-1)^n = (u-1)^n \left\{1+(u-1)\right\}^{m-n}$.

$$\text{Hence} \quad (n>r) \quad u^0 fu \left[= u^{m-n}(u-1)^n\right] = 0;$$

$$\text{and} \quad (n \text{ not} > r) \quad C_r \text{ becomes } \frac{(m-n)(m-n-1)\ldots\ldots(m-r+1)}{1 \,.\, 2 \ldots\ldots (r-n)},$$

$$\text{and} \quad u^0 fu = u^{m-n}(u-1)^n = \frac{1 \,.\, 2 \ldots\ldots r \,.\, (m-n)(m-n-1)\ldots\ldots(m-r+1)}{1 \,.\, 2 \ldots\ldots (r-n)}$$

$$= \frac{\Gamma(r+1) \,.\, \Gamma(m+1-n)}{\Gamma(m+1-r)\,\Gamma(r+1-n)}.$$

Hence, expanding,

$$u_m - \frac{n}{1} u_{m-1} + \frac{n(n-1)}{1 \,.\, 2} u_{m-2} - \&c.$$

$$\text{or} \quad [m]^r - \frac{n}{1}[m-1]^r + \frac{n(n-1)}{1 \,.\, 2}[m-2]^r - \&c.$$

$$= \frac{\Gamma(r+1)\,\Gamma(m+1-n)}{\Gamma(m+1-r)\,\Gamma(r+1-n)} \ldots\ldots\ldots\ldots\ldots\ldots (\text{II.})$$

If $n=r+1$, we obtain the proposed Formula.

(2.) If $\rho, \rho^2 \ldots\ldots \rho^{p-1}, 1$, are the p roots of unity, i. e., $\rho^p = 1$,

let $fu = (u-1)^n (u-\rho)^n (u-\rho^2)^n \ldots\ldots (u-\rho^{p-1})^n \,.\, u^{m-np}$; then $u^0 fu = u^{m-np}(u^p-1)^n$, which, as containing the factor $u^0(u-1)^n$, $= 0 \; (n>r)$.

$$\text{Hence, expanding,} \quad u_m - \frac{n}{1} u_{m-p} + \frac{n(n-1)}{1 \,.\, 2} u_{m-2p} - \&c. = 0 \quad (n>r),$$

$$\text{i. e.,} \quad [m]^r - \frac{n}{1}[m-p]^r + \frac{n(n-1)}{1 \,.\, 2}[m-2p]^r - \&c. = 0 \quad (n>r) \quad \ldots\ldots (\text{III.})$$

Ex. Let $m=\frac{1}{3}$, $n=5$, $p=7$, $r=3$, then we ought to have

$$1 \,.\, 2 \,.\, 5 + 5(20 \,.\, 23 \,.\, 26) - 10(41 \,.\, 44 \,.\, 47) + 10(62 \,.\, 65 \,.\, 68)$$
$$-5(83 \,.\, 86 \,.\, 89) + 104 \,.\, 107 \,.\, 110 = 0,$$

which is the case.

If $p=1$ and $n=r+1$ in (III.), we obtain the proposed Formula.

Note.—The preceding equations (I.), (II.), (III.), have all been obtained on the supposition of the quantity r which they involve being a positive integer. It is important to determine the true limits within which such

equations hold good. The investigation of limits, however, although interesting, is not altogether easy, and is hardly suitable for these pages. With regard to equation (II.), it may be observed that it appears to be subject to the restriction that $\sin n\pi . \sin(m-r)\pi = 0$, and further that, if n is a positive integer, m and r are quite arbitrary; if n is a negative integer, $n-r$ must not be negative; and if n is not integral, $m-r$ must be a positive integer.

1576. (Proposed by Dr. BOOTH, F.R.S.)—From n points in space let perpendiculars be drawn on a set of planes, the sum of the squares of the perpendiculars on each plane being constant; prove that these planes envelope confocal surfaces of the second order; and when the sum of the perpendiculars is constant, prove that they envelope concentric spheres.

Solution by the PROPOSER.

Let ξ, v, ζ be the tangential coordinates of one of the enveloping planes; x, y, z; x_1, y_1, z_1; &c., the projective coordinates of the n given points, and p the perpendicular from one of the n points on the plane; then we have

$$p = \frac{x\xi + yv + z\zeta - 1}{(\xi^2 + v^2 + \zeta^2)^{\frac{1}{2}}},$$

and like expressions for the other perpendiculars.

Let the sum of the squares be nk^2; then, squaring and adding, we have

$$(\Sigma x^2)\,\xi^2 + (\Sigma y^2)\,v^2 + (\Sigma z^2)\,\zeta^2 + 2\,(\Sigma yz)\,v\zeta + 2\,(\Sigma zx)\,\zeta\xi + 2\,(\Sigma xy)\,\xi v$$
$$-2\,(\Sigma x)\,\xi - 2\,(\Sigma y)\,v - 2\,(\Sigma z)\,\zeta + n = nk^2\,(\xi^2 + v^2 + \zeta^2).$$

Now let the centre of gravity of the n points be taken as the origin of coordinates, then the linear coefficients vanish; and if the principal axes of the system of n points be taken as axes of coordinates, then the coefficients of the rectangles vanish; and if, moreover, we put a, b, c for the radii of gyration round the axes of x, y, z, we shall have

$$na^2 = \Sigma x^2, \quad nb^2 = \Sigma y^2, \quad nc^2 = \Sigma z^2;$$
$$\therefore\ (k^2 - a^2)\,\xi^2 + (k^2 - b^2)\,v^2 + (k^2 - c^2)\,\zeta^2 = 1.$$

Now the differences of the squares of the semi-axes being independent of k, while a, b, c are functions of the position of the n points, the surfaces are concyclic.

NOTE.—When the sum of the perpendiculars is constant, ($\Sigma p = nk$ say,) we have

$$(\Sigma x)\,\xi + (\Sigma y)\,v + (\Sigma z)\,\zeta - n = nk\,(\xi^2 + v^2 + \zeta^2)^{\frac{1}{2}},$$

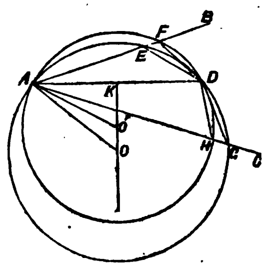
or, taking the centre of gravity of the system as origin,

$$\xi^2 + v^2 + \zeta^2 = \frac{1}{k^2};$$

which proves the second part of the theorem.—EDITOR.]

1581. (Proposed by R. PALMER, M.A.)—*If two circles pass through the vertex and a point in the bisector of an angle, prove that they intercept equal segments on the sides.*

Solution by the PROPOSER.



Let BAC be any rectilineal angle, and AD a right line bisecting it; and let any two circles AEDG, AFDH, passing through A and D, cut AB in the points E, F, and AC in G, H; then shall EF = GH.

Let O, O′ be the centres of the two circles; r_1, r_2 their radii; α, β the angles which AD makes with AO, AO′ respectively; and let $\angle \mathrm{DAB} = \theta'$, and therefore $\angle \mathrm{DAC} = -\theta'$. Then, taking AD as the initial line, we have

$\rho = 2r_1 \cos(\theta - \alpha)$ as the polar equation to the circle AEG........(1),

$\rho = 2r_2 \cos(\theta - \beta)$ as the polar equation to the circle AFH..........(2).

Now, in (1), when $\rho = \mathrm{AE}$, $\theta = \theta'$; and when $\rho = \mathrm{AG}$, $\theta = -\theta'$; hence

$$\mathrm{AE} + \mathrm{AG} = 2r_1 \left\{ \cos(\theta' - \alpha) + \cos(-\theta' - \alpha) \right\} = 4r_1 \cos\theta' \cos\alpha \ldots\ldots (3).$$

Similarly $\mathrm{AF} + \mathrm{AH} = 2r_2 \left\{ \cos(\theta' - \beta) + \cos(-\theta' - \beta) \right\} = 4r_2 \cos\theta' \cos\beta \,..\, (4).$

But from the figure we have $r_1 \cos\alpha = \mathrm{AK} = r_2 \cos\beta$;

$$\therefore\ \mathrm{AE} + \mathrm{AG} = \mathrm{AF} + \mathrm{AH}, \text{ or } \mathrm{AG} - \mathrm{AH} = \mathrm{AF} - \mathrm{AE};$$

$$\text{that is, } \mathrm{GH} = \mathrm{EF}.$$

[NOTE.—The theorem may be proved by Elementary Geometry as follows: DE = DG, and DF = DH (Euc. iii. 26, 29); also (Euc. iii. 22) $\angle$ EDG = FDH, and $\angle$ EDF = HDG; hence EF = GH (Euc. i. 4).—EDITOR.]

Memorandum on the Proof ordinarily given of the Divergency of the Harmonic Series.

By C. W. MERRIFIELD, F.R.S.

The divergency of the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

is usually proved by grouping together the series after the second term, in sets of 2, 4, 8, 16 terms, and so on; and then, since each group exceeds $\frac{1}{2}$, it is inferred that the harmonic series exceeds the series

$$1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B),$$

which is known to be divergent.

This reasoning is unsatisfactory, because it overlooks the point, that, towards the small end of the series, it requires an infinite number of terms to make up the group equal to $\frac{1}{2}$. However this point may be settled, it ought not to be ignored.

But we can easily show that it is not true at all that the series A exceeds the series B; it is not only less, but infinitely less, than the series which it is said to exceed.

In fact, the series A is the limit (for $x=1$) of $-\log(1-x)$; while the series B is the limit, also for $x=1$, of the fraction $\frac{1}{2}\frac{1}{1-x}$. Both have infinity for their limit; but it is well known that the limiting value of B is of a higher order than that of A.

The error of reasoning has probably escaped observation, owing to the known truth of the result. The divergency may be at once inferred from the equivalent expression $-\log(1-x)$, when $x=1$, becoming infinite.

Another proof (due to Mr. PURKISS) is as follows, assuming the convergency of the series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\&c. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (C),$$

which can be easily shown to be equal to $\log_e 2$.

$$\begin{aligned}\text{The series A} &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\&c.\\ &= C+2\left(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}+\tfrac{1}{8}+\&c.\right)\\ &= C+A.\end{aligned}$$

Now, as C is finite, the equation $A=C+A$ requires that A should be infinite. Hence the series A is divergent.

1530. (Proposed by M. W. CROFTON, B.A.)—Construct geometrically the expression $2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\gamma+\alpha)$.

Solution by the PROPOSER; *and* MATTHEW COLLINS, B.A.

On a circle whose radius is unity measure three consecutive arcs α, β, γ; the above is the area of the quadrilateral whose four corners are the extremities of these arcs. This appears at once from the theorem that the area of a quadrilateral is equal to that of a triangle whose vertical angle is equal to that contained by the diagonals, and the sides about it equal to the diagonals.

1521. (Proposed by J. M. WILSON, M.A.)—If any odd number of terms of a geometrical progression be taken, prove that the arithmetic mean of the odd-numbered terms is greater than the arithmetic mean of the even-numbered terms.

Solutions (1) *by* Mr. S. Bills *and the* Proposer; (2) *by* F. D. Thomson, M.A.; and Alpha.

1. Let $a, ar, ar^2 \ldots\ldots ar^{2n}$ be the series written in ascending order, so that $r>1$; then we have to show that

$$\frac{1+r^2+r^4+\ldots\ldots r^{2n}}{n+1} > \frac{r+r^3+\ldots\ldots r^{2n-1}}{n}, \text{ or } \frac{n}{n+1}\cdot\frac{r^{2n+2}-1}{r^{2n+1}-r} > 1 \ldots\ldots (a).$$

Put $r=1+s$ where s is *positive*; then (a) becomes

$$\frac{1+\frac{2n+1}{1\,.\,2}s+\frac{(2n+1)\,2n}{1\,.\,2\,.\,3}s^2+\frac{(2n+1)\,2n\,(2n-1)}{1\,.\,2\,.\,3\,.\,4}s^3+\ldots\ldots}{1+\frac{2n+1}{1\,.\,2}s+\frac{(2n+1)\,(2n-1)}{1\,.\,2\,.\,3}s^2+\frac{(2n+1)\,(2n-1)\,(2n-2)}{1\,.\,2\,.\,3\,.\,4}s^3+\ldots\ldots}>1;$$

and it is clear that the numerator of the fraction has one term more than the denominator, also that, after the first two, any term in the numerator is greater than the corresponding term of the denominator; hence the fraction is greater than unity, and the theorem is proved.

2. Otherwise; putting (a) into the equivalent form

$$f(r) \equiv nr^{2n+2}-(n+1)\,r^{2n+1}+(n+1)\,r-n>0,$$

it will be sufficient to show that the equation $f(r)=0$ has *no real root* except $r=\pm1$. Now $f(r)=0$ is evidently satisfied by $r=1$ or $r=-1$; also, forming the first and second derived equations, we see that $(r-1)$ enters as a factor in both: therefore the equation has 3 roots each equal to 1. And since there are only three changes of sign in $f(r)$ and one in $f(-r)$, the equation cannot have more than 4 real roots. Hence for all values of r *numerically* greater than unity, $f(r)$ is *positive* and >0; which proves the theorem.

1564. (Proposed by the Rev. R. H. Wright, M.A.)—Find the trilinear equations of the circles described on the sides of a triangle whose vertices are (i.) the feet of the perpendiculars from the angles of the triangle of reference on the opposite sides, (ii.) the middles of the sides, (iii.) the points in which the internal bisectors of the angles meet the opposite sides.

Solution by F. D. Thomson, M.A.

1. To prove that the equation to any circle is of the form

$$a^2yz+b^2zx+c^2xy=(p^2x+q^2y+r^2z)\,(x+y+z)$$

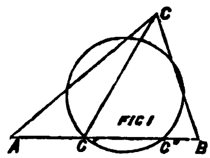
in *areal* coordinates, where p, q, r are the lengths of the tangents from the angular points of the triangle of reference.

Since all circles have a common chord at infinity, the equation to any circle must be of the form

$$a^2yz+b^2zx+c^2xy=(p^2x+q^2y+r^2z)\,(x+y+z)\ \ldots\ldots\ldots\ldots\ldots (\text{i.}),$$

where p^2, q^2, r^2 are certain constants, since $a^2yz + b^2zx + c^2xy = 0$ is the equation to the circle about the triangle of reference. It remains to determine these constants.

Let C′, C″ (Fig. 1) be the points real or imaginary in which the circle meets the side AB of the triangle of reference. Then putting $z=0$ in (i.) we get the equation



$$c^2xy = (p^2x + q^2y)(x+y),$$

or $p^2\left(\frac{x}{y}\right)^2 + (p^2+q^2-c^2)\frac{x}{y} + q^2 = 0$ .... (ii.)

to determine the positions of C′, C″. Let ρ, ρ' be the roots of (ii.) in $\frac{x}{y}$; then

$$\rho = \frac{\Delta C'BC}{\Delta C'CA} = \frac{C'B}{C'A}, \ \therefore \ \rho + 1 = \frac{c}{C'A}; \text{ similarly } \rho' + 1 = \frac{c}{C''A};$$

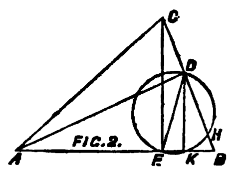
$$\therefore \ \frac{c^2}{C'A \,.\, C''A} = (\rho+1)(\rho'+1) = \rho\rho' + (\rho+\rho') + 1 = \frac{q^2 - (p^2+q^2-c^2) + p^2}{p^2} = \frac{c^2}{p^2}$$

$$\therefore \ C'A \,.\, C''A = p^2 = \text{square of tangent from A.}$$

Similarly for the other angular points.

2. To find the equation to the circle on the line joining the feet of the perpendiculars from two of the angular points on the opposite sides.

Let DHKE be such a circle described on the straight line DE as diameter, and cutting BC in H and BA in K. Then applying the theorem in Art. 1, we have



$q^2 = BD \,.\, BH$
$= (c \cos B)(EB \cos B) = ac \cos^3 B$;
$p^2 = AK \,.\, AE$
$= (AD \sin B)(b \cos A) = bc \cos A \sin^2 B$;
$r^2 = ab \cos C \sin^2 B$;

hence the *areal* equation to the circle is

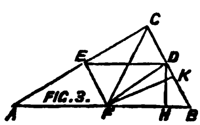
$$a^2yz + b^2zx + c^2xy = \left\{(bc \cos A \sin^2 B)\,x + (ac \cos^3 B)\,y + (ab \cos C \sin^2 B)\,z\right\}(x+y+z).$$

If the *trilinear* equation be required, write $a\alpha$ for x, $b\beta$ for y, $c\gamma$ for z, and the equation becomes

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (\alpha \cos A \sin^2 B + \beta \cos^3 B + \gamma \cos C \sin^2 B)(a\alpha + b\beta + c\gamma).$$

3. To find the equation to the circle described on the line joining the middle points of two sides of a triangle.

Let D, F (Fig. 3) be the middle points of BC, BA; join DF and draw the perpendiculars DH, FK. Then $q^2 = BF \,.\, BH = \frac{1}{4}\,ca \cos B$;



$r^2 = CD \,.\, CK = CD\left\{CD + DF \cos C\right\}$
$= \frac{1}{4}\,a(a + b\cos C)$; $p^2 = AH \,.\, AF = \frac{1}{4}\,c(c + b\cos A)$;

hence the areal and trilinear equations to the circle on FD are, respectively,

$$a^2yz + b^2zx + c^2xy = \tfrac{1}{4}\left\{c\,(c + b\cos A)\,x + (ca\cos B)\,y + a\,(a + b\cos C)\,z\right\}(x+y+z);$$

$4abc\ (a\beta\gamma + b\gamma a + c a\beta =$

$$\left\{ca\ (c a + a\gamma) + abc\ (a \cos A + \beta \cos B + \gamma \cos C)\right\} (a a + b\beta + c\gamma)$$

4. To find the equation to the circle on the line joining the points where the bisectors of two angles meet the opposite sides.

Let D, F (Fig. 3) be the two points. Then $q^2 = \mathrm{BF} \,.\, \mathrm{BH} = \mathrm{BF} \,.\, \mathrm{BD} \cos B$;

$$\text{but } \frac{\mathrm{AF}}{\mathrm{FB}} = \frac{b}{a}, \therefore \mathrm{BF} = \frac{ca}{b+a}; \text{ and similarly } \mathrm{BD} = \frac{ca}{c+b};$$

$$\therefore q^2 = \frac{c^2a^2 \cos B}{(a+b)\,(b+c)}.$$

$$\text{Again } r^2 = \mathrm{CK} \,.\, \mathrm{CD} = \left(a - \frac{ca \cos B}{b+a}\right) \frac{ab}{c+b} = \frac{a^2b^2\,(1+\cos C)}{(a+b)\,(b+c)};$$

$$\text{and similarly } p^2 = \frac{b^2c^2\,(1+\cos A)}{(b+c)\,(c+a)}.$$

Hence the areal and trilinear equations to the circle are, respectively,

$${}^2yz + b^2zx + c^2xy =$$

$$\frac{b^2c^2(a+b)(1+\cos A)x + c^2a^2 \cos B(c+a)\,y + a^2b^2(a+b)(1+\cos C)z}{(a+b)\,(b+c)\,(c+a)}\,(x+y+z);$$

$$(a+b)\,(b+c)\,(c+a)\,(a\beta\gamma + b\gamma a + c a\beta) = \Big\{bc\ (a+b)\ (1+\cos A)\ a +$$
$$ca \cos B\ (c+a)\,\beta + ab\,(a+b)\ (1+\cos C)\ \gamma\Big\}\ (a a + b\beta + c\gamma).$$

1058. (Proposed by Exhumatus.)—A straight piece of wire is bent at random into two arms and then suspended by an extremity. Find the probability that the angle will, in the position of equilibrium, rise above the point of suspension.

Solution by the Editor.

Taking the length of the rod as unity, let x be the length of the *free* arm, and θ the angle contained by the two arms; then, by taking moments about the point of suspension, we find that the upper arm will be horizontal if $x^2 = \frac{1}{2} \sec^2 \frac{1}{2}\theta$; hence the probability that the vertex of the angle will be *above* the point of suspension is $(1 - \frac{1}{2}\sqrt{2} \,.\, \sec \frac{1}{2}\theta)$ when $\theta < \frac{1}{2}\pi$, and *zero* when $\theta > \frac{1}{2}\pi$. Now the angle contained by the arms may, with equal probability, have any value from 0 to π; so that the probability that it will be between θ and $\theta + \Delta\theta$ is $(\Delta\theta : \pi)$; hence, putting p for the required probability, we have

$$p = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} (1 - \tfrac{1}{2}\sqrt{2} \,.\, \sec \tfrac{1}{2}\theta)\, d\theta = \frac{1}{\pi}\Big[\theta - \sqrt{2} \,.\, \log \tan \tfrac{1}{4}(\pi + \theta)\Big]_{\theta=0}^{\theta=\frac{1}{2}\pi}$$

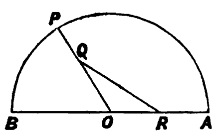
$$= \frac{1}{2} - \frac{\sqrt{2}}{\pi} \log \tan \tfrac{3}{8}\pi,$$

$$\text{or } p = \frac{1}{2} - \frac{\sqrt{2}}{\pi} \log_e (1 + \sqrt{2}).$$

The value of p is ·10325 nearly; and the fractions converging to this value are $\frac{1}{9}$, $\frac{1}{10}$, $\frac{3}{29}$, $\frac{16}{155}$, &c.; hence the odds are nearly 9 to 1 *against* the angle being *above* the point of suspension.

1160. (Proposed by EXHUMATUS.)—Given the sum of two sides of a triangle, and nothing else; find the mean value of the third side.

Solution by the EDITOR.



Let APB be a semicircle, the radius OA of which is equal to the given sum of the sides (OQ + OR) of the triangle OQR, and let it be required to find the mean value of the third side QR.

Put $OQ = r$, $\angle QOA = \theta$, and $OA = OQ + OR = 2a$, then we have

$$QR^2 = r^2 - 2r(2a-r)\cos\theta + (2a-r)^2 = 4a^2 - 4r(2a-r)\cos^2\tfrac{1}{2}\theta.$$

Now it is clear that for every point within the semicircle (but for no point outside it) there is a position of Q and a corresponding value of QR which will satisfy the conditions of the problem; and as the mean value of a function is the quotient obtained by dividing the sum of all its values by the number of such values, which may in this case be measured by the area of the semicircle, we have, putting μ for the mean value required,

$$\mu = \frac{\Sigma\,(QR)}{\text{Area APB}} = \frac{1}{\pi a^2}\int_0^{\pi}\!\!\int_0^{2a} \left\{a^2 - r(2a-r)\cos^2\tfrac{1}{2}\theta\right\}^{\frac{1}{2}} r\,d\theta\,dr \;\ldots\ldots\ldots\ (1).$$

This expression for μ may be otherwise obtained by taking the average of n values of QR, and then finding the limit of the fraction when n is increased without limit; thus

$$\mu = \frac{(QR)_1 + (QR)_2 + \ldots\ (QR)_n}{n} = \frac{\left\{(QR)_1 + (QR)_2 + \&c.\right\} r\,\Delta\theta\,\Delta r}{nr\,\Delta\theta\,\Delta r};$$

$r\,\Delta\theta\,\Delta r$ being a sectorial element at (r, θ). The limit of the denominator will be the area of the semicircle (viz., $2\pi a^2$), and that of the numerator is

$$\int_0^{\pi}\!\!\int_0^{2a} 2\left\{a^2 - r(2a-r)\cos^2\tfrac{1}{2}\theta\right\}^{\frac{1}{2}} r\,d\theta\,dr;$$

hence we obtain for μ the same value (1) as before.

Let $r=\rho+a$, then (1) becomes

$$\pi a^2\mu=\int_0^\pi\int_{-a}^a(\rho^2+a^2\tan^2\tfrac{1}{2}\theta)^{\frac{1}{2}}(\rho+a)\cos\tfrac{1}{2}\theta\,d\theta\,d\rho\ldots\ldots\ldots\ldots\ldots(2).$$

But $\int_{-a}^a(\rho^2+a^2\tan^2\frac{1}{2}\theta)^{\frac{1}{2}}\rho\,d\rho=0$; hence we have

$$\mu=\int_0^\pi\frac{d\theta}{\pi}\int_0^a\frac{2}{a}\cos\tfrac{1}{2}\theta\,(\rho^2+a^2\tan^2\tfrac{1}{2}\theta)^{\frac{1}{2}}d\rho\ldots\ldots\ldots\ldots\ldots(3).$$

Again, $\int_0^a 2\,(\rho^2+a^2\tan^2\frac{1}{2}\theta)^{\frac{1}{2}}\,d\rho=a^2(\sec\frac{1}{2}\theta+\tan^2\frac{1}{2}\theta\log\cot\frac{1}{4}\theta)$;

$$\therefore\ \mu=\int_0^\pi\frac{d\theta}{\pi}\left\{a+a(\sec\tfrac{1}{2}\theta-\cos\tfrac{1}{2}\theta)\log\cot\tfrac{1}{4}\theta\right\}\ldots\ldots\ldots\ldots(4).$$

Also, $\int_0^\pi d\theta\cos\frac{1}{2}\theta\log\cot\frac{1}{4}\theta=\int_0^\pi 2\log\cot\frac{1}{4}\theta\,d(\sin\frac{1}{2}\theta)=\int_0^\pi d\theta=\pi$;

and if we put $\cot\frac{1}{4}\theta=e^x$, we shall have

$$\int_0^\pi d\theta\sec\tfrac{1}{2}\theta\log\cot\tfrac{1}{4}\theta=\int_0^\infty 4x\,(e^{-x}+e^{-3x}+e^{-5x}+\ldots)\,dx$$
$$=4\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)=\frac{\pi^2}{2};$$

hence, finally, we have $\mu=\frac{1}{2}\pi a$.

The second factor on the right-hand side of (3) or (4) expresses by itself the mean value of the third side of a triangle when the sum ($2a$) of the other two sides is given, and also the angle (θ) which they contain.

For suppose AOP to be the given angle (θ); and on OA, OP let n segments (h) be taken, where $nh=2a$; then, if the alternate points of section be joined, we shall have n values of QR such that $\text{OQ}+\text{OR}=2a$; and the average of these n values is

$$\frac{\Sigma(\text{QR})}{n}=\frac{h\Sigma(\text{QR})}{2a};$$

consequently, when n is increased without limit, this average will be

$$\int_0^{2a}\frac{(\text{QR})\,dr}{2a}=\int_0^a\frac{2}{a}\cos\tfrac{1}{2}\theta\,(\rho^2+a^2\tan^2\tfrac{1}{2}\theta)^{\frac{1}{2}}d\rho$$
$$=a+a(\sec\tfrac{1}{2}\theta-\cos\tfrac{1}{2}\theta)\log\cot\tfrac{1}{4}\theta\ldots\ldots(5).$$

For example, if the sum of the two sides of a *right-angled* triangle be constant ($=2a$), the mean value of the hypotenuse is, by (5),

$$\left\{1+\tfrac{1}{2}\sqrt{(2)}\log(1+\sqrt{2})\right\}a,\ \text{ or }\ \tfrac{8}{5}a\ \text{nearly.}$$

Again, suppose that θ varies continuously from 0 to π; then the mean value (μ) required in the Question will be the average of the resulting values of the expression (5); thus we have, as before,

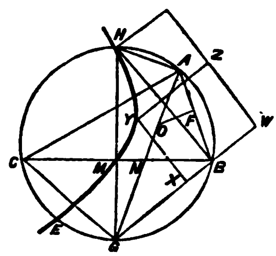
$$\mu=\int_0^\pi\frac{d\theta}{\pi}\left\{a+a(\sec\tfrac{1}{2}\theta-\cos\tfrac{1}{2}\theta)\log\cot\tfrac{1}{4}\theta\right\},$$

$$\therefore\ \mu=\tfrac{1}{2}\pi a,\ \text{ or }\ \tfrac{11}{7}a\ \text{nearly.}$$

1570. (Proposed by Mr. W. H. LEVY.)—Given the difference between the base and the sum of the sides of a triangle, the diameter of the circumscribed circle, and the line bisecting the vertical angle and terminating in the circumference of this circle; to determine the triangle.

Solution (I.) *by the* PROPOSER; (II.) *by* MR. D. M. ANDERSON.

I. CONSTRUCTION. Take GW equal to the given line bisecting the vertical angle, and draw WZ perpendicular to GW. Construct the square WXYZ equal to the *given* rectangle D . D_1, where D is equal to the *given* diameter of the circumscribed circle, and D_1 to half the *given* difference between the sum of the sides and the base. With centre W, vertex Y, and asymptotes WG and WZ, construct the rectangular hyperbola HYE. Also with centre G and radius equal to the *given* diameter D, describe an arc intersecting the hyperbola in H. Join GH, on which describe the circle GBH. Also with centre G and radius equal to GW describe an arc cutting the circle in A. Draw BC perpendicular to GH and meeting the circle in C; then will ABC be the triangle required.



DEMONSTRATION. Join GA, and BH. In GA take GO = GB, and draw OF perpendicular to AB. By construction, the diameter GH = D the *given* diameter; also AG, bisecting the angle BAC, is equal to the given bisecting line; and AO = BW. Again, by similar triangles AOF, HGB, we have AO (or BW) : AF = HG : HB; ∴ AF . GH = BW . BH. But by the property of the hyperbola BW . BH = WX^2 = D . D_1 = the given rectangle, ∴ AF = D_1 = as is well known to half the difference between the sum of the sides AB, AC, and the base BC.

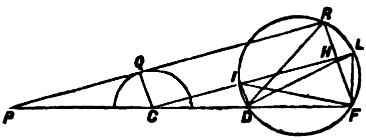
II. Otherwise; let the diameter HG (or D) and the bisector AG (or l) meet the base BC (or a) in M, N respectively; then since CG = GB, we have CG (AB + AC) = BC . AG, whence $CG = \dfrac{al}{AB+AC} = \dfrac{al}{a+2D_1}$;

$$\frac{a^2l^2}{(a+2D_1)^2} = CG^2 = HG \cdot GM = \tfrac{1}{2}D\left[D \pm \sqrt{(D^2-a^2)}\right].$$

From this equation a can be found in terms of the known quantities l, D, D_1; and the other parts of the triangle may be readily determined therefrom.

1571. (Proposed by Mr. J. CONWILL.)—Draw a straight line touching a given semicircle in Q, and meeting the diameter produced, and a line given in position, in P, R, so that PQ : QR = a given ratio.

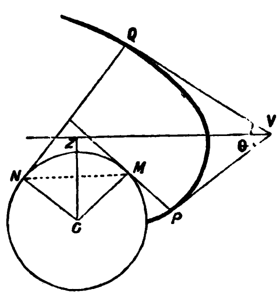
Solution by MR. W. HOPPS, *and* MR. P. W. FLOOD.



Let C be the centre of the given semicircle, and D the intersection of its diameter with the line given in position. Draw RF perpendicular to PR, meeting CD at F; also through C draw a line parallel to PR, meeting RF at H; then PQ : QR = CQ : HF, hence HF and RF are given: but the angle RDF is given, therefore the circle through R, D, F is given in magnitude. Let I, L be the intersections of this circle with CH; then the chord IL is obviously given, and $\angle$ IDR = IFR, which is given; also $\angle$ RDL = RFL, which is given; but RD is given in position, therefore DI, DL are lines given in position; and as IL is given in length, we have to draw from the given point C a line cutting the lines DI, DL (given in position) in I, L, so that IL shall be of a given length; which is a very old problem of some celebrity.

1582. (Proposed by M. W. CROFTON, B.A.)—Two tangents to the involute of a circle contain a given angle; prove that the straight line bisecting their angle always touches a fixed circle, concentric with the generating circle.

Solution by the PROPOSER.



Let VP, VQ be two tangents to the involute, including a given angle (θ); draw perpendiculars to them (PM, QN) touching the generating circle in M, N; also draw VZ bisecting the angle V, and CZ perpendicular to VZ from the centre C.

Now if p, p', P be the perpendiculars from any point on the sides of an angle (θ) and on the bisector of that angle, it is easily shown that $2P \cos \frac{1}{2}\theta = p - p'$.

Here the perpendiculars from C on the sides of the angle PVQ are equal to PM, QN; and that on the bisector is CZ;

$$\therefore\ 2CZ \cos \tfrac{1}{2}\theta = QN - PM = \text{arc } MN.$$

But MN subtends an angle $\pi - \theta$ at C, and is therefore given;

$$\text{and } CZ = \frac{\text{arc } MN}{2 \cos \frac{1}{2}\theta} = \text{radius} \times \frac{\text{arc } MN}{\text{chord } MN},$$

so that CZ is a given quantity; hence the bisector always touches a circle whose radius is thus found, and whose centre is C.

1523. (Proposed by the EDITOR.)—If $[x]^n$ denote the factorial expression

$$x(x-1)(x-2)\ldots\ldots(x-n+1),$$

show how to interpret $[\pm x]^0$ and $[\pm x]^{-n}$.

Solution by H. R. GREER, B.A.

Denoting the factorial $x(x-1)\ldots\ldots(x-n+1)$ by x_n, we may justly define x_{-n} to mean $\dfrac{1}{(x+1)(x+2)\ldots\ldots(x+n)}$. For, with this notation, it can be shown that whatever *general* and formal laws x_n obeys for positive values of n, it obeys also for negative values. As for instance, $\Delta x_n = n\Delta x_{n-1}$, also $x_n = \dfrac{\Gamma(x+1)}{\Gamma(x+1-n)}$, both which hold for positive *and* negative values of the suffix. There is a third law obtaining for general values of n in this notation, which I wish to prove and to consider, very briefly, in some of its consequences. Let us denote, as usual, the operation of differentiating with regard to x by the symbol D_x, (the suffix being sometimes omitted as unnecessary); it is easy to derive from the fundamental law, viz., $Dx = xD + 1$, the symbolical theorems, $x^nD^n = xD(xD-1)\ldots\ldots(xD-n+1)$, and $D^nx^n = (xD+1)(xD+2)\ldots\ldots(xD+n)$; that is to say, using factorial notation, $(xD)_n = x^nD^n$; and $\{(xD)_{-n}\}^{-1} = D^nx^n = \{x^{-n}D^{-n}\}^{-1}$, therefore $(xD)_{-n} = x^{-n}D^{-n}$, thus establishing the formula $(xD)_n = x^nD^n$ as true for both positive and negative values of n. Hence we have the symbolical theorem $x^nD^n = \dfrac{\Gamma(xD+1)}{\Gamma(xD+1-n)}$. Let us now, by way of experiment, try the effect of giving to n fractional values; premising that we shall discard all results that do not admit of some verification. Assume, then, $x^{\frac{1}{2}}D^{\frac{1}{2}} = \dfrac{\Gamma(xD+1)}{\Gamma(xD+\frac{1}{2})}$, and operate with each of these forms upon x^m by help of the theorem $f(xD)\,.\,x^m = f(m)\,.\,x^m$, we obtain $x^{\frac{1}{2}}D^{\frac{1}{2}}\,.\,x^m = \dfrac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}x^m$, and therefore $D^{\frac{1}{2}}\,.\,x^m = \dfrac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}x^{m-\frac{1}{2}}$. We may now verify this result by repeating upon both sides of it the operation $D^{\frac{1}{2}}$, according to the rule which itself supplies. At first sight it would seem that the only restriction upon the formula need be that $m+\frac{1}{2}$ must be positive; if, however, *any* negative value (however small) be assigned to m, the above-given verification becomes impossible. This formula, therefore, of fractional differentiation cannot be assumed to hold except for positive exponents of the variable. The question of the same operation for inverse powers of the variable has been treated in the *Quarterly Journal of Mathematics*, vol. iv., by M. Wastchenxo-Zachartenxo, who has obtained the result $D^{\frac{1}{2}}\,.\,\dfrac{1}{x^r} = \sqrt{(-1)}\dfrac{\Gamma(r+\frac{1}{2})}{\Gamma(r)}\,.\,\dfrac{1}{x^{r+\frac{1}{2}}}$; which holds for positive values of r only. Particular cases worthy of remark

are $D^{\frac{1}{2}}x = \frac{2}{\sqrt{(\pi)}} \cdot \sqrt{(x)}$, and $D^{\frac{1}{2}} \cdot \frac{1}{x} = \sqrt{(-1)}\frac{\sqrt{(\pi)}}{2} \cdot \frac{1}{x^{\frac{3}{2}}}$. But I will not pursue this topic further at present. Vandermonde's theorem, viz.,

$$(x+h)_m = x_m + m\,(x)_{m-1} \cdot h + \frac{m\,(m-1)}{1\,.\,2}\,(x)_{m-2} \cdot (h)_2, \text{ \&c.}$$

holds of course for negative values of m, and may be proved by operating with $(1+\Delta)^h$ upon x_{-m}. I may remark that the same theorem furnishes an instantaneous proof of a useful symbolical theorem, due to the late much lamented Rev. R. Carmichael; viz., denoting by ∇ the complex operation $xD_x + yD_y +$ &c., and by ∇_p the development of ∇ by the Binomial Theorem, in each term of the development the symbols of operation being transposed entirely to the right hand, and those of quantity to the left; then $\nabla_p = \nabla \,.\, (\nabla - 1) \ldots\ldots (\nabla - p + 1)$. In fact, the right-hand member of this equation is what is denoted in factorial notation by ∇_p; the development of this, by Vandermonde's theorem, consists of a number of terms of the form (neglecting the coefficient) $(xD_x)_r \cdot (yD_y)_s \ldots$, *i. e.*, of the form $x^r D_x^r \cdot y^s D_y^s \ldots$, *i.e.*, of the form $x^r y^s \ldots D_x^r D_y^s \ldots$, which is the definition in this case of the development of ∇_p; in other words, ∇_p, according to one definition, $= \nabla_p$, according to the other.

1485. (Proposed by R. TUCKER, M.A.)—In two parallel planes (A, B) are taken m and n points respectively, no three of which are in the same straight line, with the exception of p of the A-points, and q of the B-points, which lie in straight lines; find (1) the number of triangles which can be formed by joining all the points in any manner, (2) the number of triangular pyramids with their bases in the planes.

Solution by the PROPOSER.

Let ${}_nC_r$ denote the number of combinations of n things taken r at a time; then in the plane (A) the number of triangles will be

$${}_{m-p}C_3 + p\,{}_{m-p}C_2 + (m-p)\,{}_pC_2 = \text{M}.$$

Similarly the number in the plane (B) will be

$${}_{n-q}C_3 + q\,{}_{n-q}C_2 + (n-q)\,{}_qC_2 = \text{N}.$$

For the combination of (A) with (B) we have

$$n\,{}_mC_2 + m\,{}_nC_2 = \text{P}.$$

Hence the whole number of triangles is M + N + P.
The number of triangular pyramids will be nM + mN.

1603. (Proposed by the Rev. J. BLISSARD.)—If $T_1, T_2, T_3 \ldots . T_n$ be the sum of the products of the n quantities, $\tan x$, $\tan 2x$, $\tan 2^2x, \ldots . \tan 2^{n-1}x$, taken 1, 2, 3, ... n together; prove that

$$(1.)\quad 1-T_2+T_4-T_6+\&c. = 2^n \sin x \cos(2^n-1)\,x \operatorname{cosec}(2^n x),$$

$$(2.)\quad T_1-T_3+T_5-\&c. = 2^n \sin x \sin(2^n-1)\,x \operatorname{cosec}(2^n x).$$

Solution by R. TUCKER, M.A.; *and others.*

Assume $u+iv = (1+i\tan x)(1+i\tan 2x)\ldots\ldots(1+i\tan 2^{n-1}x)$; then, equating rational and irrational parts, we have

$$u = 1-T_2+T_4-T_6+\&c. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i.)}$$

$$\text{and } v = T_1-T_3+T_5-T_7+\&c. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii.)}$$

Again, $(u+iv)\cos x\cos 2x\ldots.\cos 2^{n-1}x$

$$= (\cos x+i\sin x)(\cos 2x+i\sin 2x)\ldots\ldots$$

$$= \cos(1+2+2^2+\ldots.2^{n-1})\,x+i\sin(1+2+\ldots.2^{n-1})\,x$$

$$= \cos(2^n-1)\,x+i\sin(2^{n+x}-1)\,x.$$

Now we may easily show that

$$2^n\sin x(\cos x\cos 2x\cos 2^2x\ldots.\cos 2^{n-1}x) = \sin 2^n x;$$

$$\therefore\ u+iv = 2^n\sin x\operatorname{cosec}2^n x\left\{\cos(2^n-1)\,x+i\sin(2^n-1)\,x\right\};$$

$$\therefore\ \text{again } u = 2^n\operatorname{cosec}2^n x\sin x\cos(2^n-1)\,x \quad\ldots\ldots\ldots\ldots \text{(iii.)}$$

$$\text{and } v = 2^n\operatorname{cosec}2^n x\sin x\sin(2^n-1)\,x \quad\ldots\ldots\ldots\ldots \text{(iv.)}$$

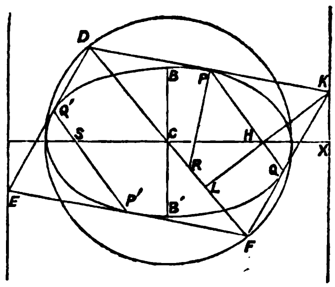
Equating (i.) with (iii.) and (ii.) with (iv.), we get the formulæ required.

[Or the particular case may be readily inferred from the general formulæ in Todhunter's *Trigonometry*, Arts. 129, 273.]

1544. (Proposed by R. TUCKER, M.A.)—Tangents to an ellipse are drawn at the extremities of pairs of parallel focal chords; prove that the parallelograms thus formed vary inversely as the projection of the chords on the minor axis; also find the conditions of a maximum or minimum area.

Solution by the PROPOSER.

Let DEFK be one of the parallelograms circumscribed to the ellipse at the extremities of a pair of parallel focal chords PHQ, P′SQ′; then the



diagonals intersect in the centre C of the ellipse. By known properties, E, K lie on the directrices, and D, F on the auxiliary circle. Join KH, and produce it to meet the diameter DCF in L; then, by another known property, KH is perpendicular to PQ, and therefore KL perpendicular to DF. Hence, putting $\angle$CHP$=\theta$, we have

$$\text{area of DEFK} = \text{DF} \,.\, \text{KL}$$

$$= 2a\left\{ae \sin\theta + \left(\frac{a}{e} - ae\right)\operatorname{cosec}\theta\right\}$$

$$= \frac{2a^2(1-e^2\cos^2\theta)}{e\sin\theta} = \frac{4ab^2}{e(\text{PQ}\sin\theta)},$$

which varies *inversely* as the projection (PQ $\sin\theta$) of PQ on the minor axis (BCB′).

[The area of the *inscribed* parallelogram PQP′Q′ (= PQ . HL = PQ . CH $\sin\theta$) varies *directly* as the projection of PQ on the minor axis; it is therefore a maximum when DEFK is a minimum, and *vice versâ*. Moreover BC is a mean proportional between PH and LF, or between DL and HQ; also KL . LH = DL . LF, which in fact follows at once from the equality of the angles KFL, FDE, DHL, the last two of which are formed by two pairs of lines mutually perpendicular. Again, if the normal at P meet DF in R, SP will have to PH the duplicate ratio of PR to BC; whence, by a known property (viz., SP : PH = SD^2 : BC^2), it follows that PR = SD. Suppose DS, PR to meet EF in T, V (T being on the auxiliary circle); then RV = ST, and therefore PR . RV = DS . ST = BC^2; also DR = SP, RF = PH, DR . RF = SP . PH = square of semi-diameter conjugate to CP; and HF = PR = DS.]

The maximum or minimum values of the area depend upon those of

$$u = (1-e^2\cos^2\theta)\operatorname{cosec}\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

whence $$\frac{du}{d\theta} = (e^2-1+e^2\sin^2\theta)\cos\theta\operatorname{cosec}^2\theta \ldots\ldots\ldots\ldots\ldots (2),$$

and $$\frac{d^2u}{d\theta^2} = (1-e^2)(2\operatorname{cosec}^2\theta-1)\operatorname{cosec}\theta - e^2\sin^2\theta \ldots\ldots\ldots (3).$$

From the symmetry of the figure we readily see that we need only discuss values of θ ranging from 0 to $\frac{1}{2}\pi$. We see also that u increases without limit as θ approaches to zero, that is, as the chord approaches to coincidence with the major axis.

The values of θ which make (2) vanish are

$$\theta = \tfrac{1}{2}\pi \ldots\ldots\ldots\ldots (\alpha); \quad \sin\theta = \frac{\sqrt{(1-e^2)}}{e} = \frac{b}{ae} \ldots\ldots\ldots\ldots (\beta).$$

With the value (α), we get (3) $= 1-2e^2$; hence there is, in this case, a maximum if $e^2 > \frac{1}{2}$, and a minimum if $e^2 < \frac{1}{2}$.

To get a maximum or minimum value from (β), we must have (since $\sin\theta$ is a proper fraction) $e^2 > 1-e^2$, or $e^2 > \frac{1}{2}$; and as (3) becomes in this case the *positive* quantity $2(2e^2-1)\operatorname{cosec}\theta$, we have a minimum.

[It will be readily seen that the minimum-giving position of the chord (when $e^2 > \frac{1}{2}$) is obtained by drawing from the focus a tangent to the circle on the minor axis as diameter. The minimum area then $\left(\text{when } \theta = \sin^{-1}\frac{b}{ae}\right)$ is $4ab$; and the area when the chord coincides with the parameter $(\theta = \frac{1}{2}\pi)$ is $\frac{2a^2}{e}$.]

II. *Solution by* F. D. THOMSON, M.A.

Let ϕ, ϕ', $\pi+\phi'$ be the eccentric angles of the points P, Q′, Q; then the equations of the tangents at P, Q′, Q are

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1; \quad \frac{x}{a}\cos\phi' + \frac{y}{b}\sin\phi' = 1; \quad \frac{x}{a}\cos\phi' + \frac{y}{b}\sin\phi' = -1;$$

hence, putting (x', y'), (x'', y'') for the coordinates of D, K, we have

$$x' = a\frac{\sin\phi' - \sin\phi}{\sin(\phi'-\phi)}, \quad y' = b\frac{\cos\phi' - \cos\phi}{\sin(\phi-\phi')};$$

$$x'' = a\frac{\sin\phi' + \sin\phi}{\sin(\phi'-\phi)}, \quad y'' = b\frac{\cos\phi' + \cos\phi}{\sin(\phi-\phi')};$$

$$\therefore\ \text{DK}^2 = (x'-x'')^2 + (y'-y'')^2 = \frac{4(a^2\sin^2\phi + b^2\cos^2\phi)}{\sin^2(\phi'-\phi)}.$$

But the perpendicular (CZ suppose) from C on DK is given by

$$\text{CZ}^2 = \frac{a^2b^2}{a^2\sin^2\phi + b^2\cos^2\phi};$$

$$\therefore \text{ area of parallelogram DEFK} = 2\text{CZ}.\text{DK} = \frac{4ab}{\sin(\phi'-\phi)}.$$

But K is on the directrix, since PQ is a focal chord;

$$\therefore\ x'' = \frac{a}{e}, \quad \text{whence} \quad \sin(\phi'-\phi) = e(\sin\phi' + \sin\phi);$$

$$\therefore\ \text{DEFK} = \frac{4ab}{e(\sin\phi + \sin\phi')} = \frac{4ab^2}{e\,(\text{proj. of PQ on minor axis})}.$$

The rest follows as in the foregoing Solution.

[It is readily seen that Mr. Thomson's investigation is equally applicable whether e denote the eccentricity or not: e may, in fact, be any proper fraction; that is to say, the property holds if the parallels PHQ, P′SQ′ are drawn through any two points H, K in the major axis equidistant from the centre.

The locus of K is, of course the polar of H; moreover, by eliminating ϕ, ϕ' from the equations $x' =$ &c., $y' =$ &c., $\sin(\phi'-\phi) =$ &c., which is easily done, we find that the locus of D is the ellipse

$$\frac{x^2}{a^2} + (1-e^2)\frac{y^2}{b^2} = 1;$$

and when e is the eccentricity of the given ellipse, that is to say, when H, S are the foci, the equation becomes

$$x^2 + y^2 = a^2,$$

which is that of the circle circumscribing the ellipse.]

1573. (Proposed by Dr. HIRST, F.R.S.)—In the system of conics which can be inscribed in a given triangle so that the normals at the points of contact are concurrent; how many are there which touch a given conic?

Solution by the PROPOSER.

In my Solution of Quest. 1545, from which the present question originated, it was proved that there are, in the system under consideration, *three* conics which touch any assumed line. It will, moreover, be presently shown that there are, in general, *six* conics of the system which pass through any assumed point. A knowledge of these two numbers, 6 and 3, which by M. Chasles are termed the *characteristics* of the system, and represented, generally, by μ and ν, enables us to determine, completely, all properties of the system. For instance, the answer to our present question is simply twice the sum of these characteristics, or *eighteen*; and that in virtue of the following three theorems, given by Chasles in his recent important contributions to the Theory of Conics.

(α) The poles of a given right line, relative to the several conics of any system (μ, ν), are situated on a curve of the order ν.

(β) The polars of a given point, relative to the several conics of any system (μ, ν), envelope a curve of the class μ.

(γ) The locus of a point which has the same polar, relative to a given conic C, as it has with respect to any conic of a system (μ, ν) is a curve of the order $(\mu+\nu)$.

The first and second of these theorems are mutually correlative; and the first is evident from the fact that the locus in question cannot cut the given line in more than ν points. In fact, each point of intersection of the locus and the given line being a pole of that line relative to some conic of the system, the latter must there touch the line; but by hypothesis there are only ν conics which do so.

The theorem (γ) may be thus established. On any line L whatever take any point a, and find its polar A relative to the given conic C. According to (α) there are ν points a' on L whose polars, relative to ν conics of the system (μ, ν), coincide with A. To prove our theorem, we require to know how often two such corresponding points a, a' coincide. Now by (β) there are, relative to conics of the system, μ polars A of a', which pass through the pole l, relative to C, of the line L, and each of these polars has, of course, its pole a, relative to C, situated on L. Hence, the relation between the points a, a' is such that to each point a correspond ν points a', whilst to each point a' there correspond μ points a. The distances x, x' of a and a' from any origin on L must, consequently, be connected by an equation of the μth degree in x, and of the νth in x'. This granted, the condition $x = x'$ must lead to an equation of the $(\mu+\nu)$th degree, whose roots will determine the points in which L intersects the required locus. The latter, therefore, is of the order stated in the theorem.

This locus intersects any given *arbitrary* conic C in $2(\mu+\nu)$ points, each of which has for its polar, relative to a conic of the system, the tangent thereat to C. Hence we conclude, as above stated, that in the system (μ, ν) there are, in general, $2(\mu+\nu)$ conics which touch a given conic.

It is scarcely necessary to remark, that this result will suffer modification, if the given conic, instead of being perfectly arbitrary, have any special relation to the system.

The above characteristic 6, or the number of conics of our system which

pass through any assumed point, results from the following theorems, which have also an interest of their own :—

(1) If a conic be inscribed in a triangle ABC so that the connectors of the three points of contact α, β, γ, with three fixed points D, E, F, respectively, are concurrent, the locus of the point of concurrence will be a cubic circumscribed to ABC and DEF, and passing through the intersections

(BF, CE), (CD, AF), (AE, BD).

The connectors Aα, Bβ, Cγ, as is well known, are always concurrent; if then any one Aα be fixed, the two others will generate homographic pencils, and the points of contact β, γ homographic ranges. The connectors Eβ, Fγ, therefore, will also describe homographic pencils, and the intersections of their corresponding elements will lie on a conic $(E, F)^2$ passing through the centres of the pencils, as well as through the corner A and the intersection (BF, CE). This conic will manifestly intersect Dα in two points of the required locus. Thus to each point α, and consequently to each ray Dα, will correspond a determinate conic $(E, F)^2$ passing through four fixed points, and *vice versâ* to each element of this pencil of conics will correspond *one* easily constructed ray Dα. According to known principles, therefore, the pencil of rays and the pencil of conics correspond anharmonically, and generate, by the intersection of their corresponding elements, a cubic passing through the centre D of the linear pencil, and the four basic points E, F, A, (BF, CE) of the quadric pencil. This is obviously the required locus; for the latter being necessarily symmetrical with respect to the triangles ABC and DEF must pass through the remaining four points B, C, (CD, AF), (AE, BD) alluded to in the theorem.

(2) In the system of conics inscribed to a triangle ABC, and passing through a fixed point P, there are two conics which touch any side BC of that triangle in a given point α.

It has already been shown that the points of contact β, γ describe homographic ranges on AC and AB, and since two corresponding points of the latter coincide in A, and B and C also correspond to each other, we conclude that the connector $\beta\gamma$ always passes through a fixed point α' on BC. This point α' is in fact the pole of Aα relative to every inscribed conic which touches at α, and it is likewise the harmonic conjugate of α relative to B and C. From this it follows that, if any inscribed conic touching at α pass through the point P, it must likewise pass through Q, the fixed harmonic conjugate of P relative to α', and the intersection (α'P, αA). Now the conics which pass through P and Q and touch BC in α cut AB in pairs of points in involution, and the double points of the latter are the points of contact of the only two conics in such a series which touch AB. But touching AB each of these conics will also touch AC, since α' and Aα are pole and polar with respect to it. The theorem, therefore, is established.

(3) If a conic be inscribed to a triangle ABC so as to pass through a given point P, the locus of the intersection of the connectors of the points of contact β, γ on any two sides of the triangle with two fixed points E, F, respectively, will be a quartic having a double point at the intersection A of those sides, as well as at each of the fixed points E and F.

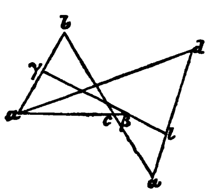
Let EF intersect the side AC, in β, then by (2) there are two inscribed conics which touch AC at β and likewise pass through P; their points of contact γ, γ' on AB being connected with F, give two intersections with Eβ coincident with F; that is to say, F must be a double point on the required locus. The fixed point E will, for similar reasons, be another double point. Again, if any line whatever through E cut AC in β, and

γ, γ' be the points of contact, on AB, of the two inscribed conics through P which touch AC at β, the connectors $F\gamma$, $F\gamma'$ will intersect $E\beta$ in the only two points of the locus, exclusive of the double point E, which $E\beta$ can contain; in other words, the locus required is of the fourth order. On the line EA the two points of the quartic, which are usually distinct, obviously coincide in A, and the same is true of the two points, exclusive of F, on FA. The quartic, therefore, as stated in the theorem, has a third double point at A.

Exclusive of the double points A, E, F, this quartic intersects the above cubic (1) in *six* points, which are easily recognized to be the only points in the plane in each of which intersect all three connectors $D\alpha$, $E\beta$, $F\gamma$ relative to a conic inscribed to ABC so as to pass through P. In other words, in the system of conics inscribed to the triangle ABC so that the connectors of the points of contact α, β, γ with three fixed points D, E, F, respectively, are concurrent, there are *six* curves which pass through any assumed point P. The points D, E, F in our Question are at infinity in directions perpendicular to the sides of ABC; the characteristics of the system of conics in our Question are the same as those of the more general system just considered.

1607. (Proposed by Professor CAYLEY.)—In a given cubic curve to inscribe a triangle such that the three sides shall pass respectively through three given points on the curve.

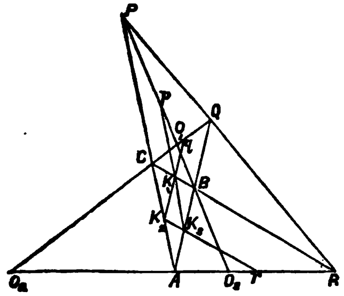
Solution by PROFESSOR CREMONA.



Let abc be any triangle inscribed in a cubic, and let its sides bc, ca, ab cut the curve again in α, β, γ, respectively. Then, if the tangent at a cut the curve in d, and l be the third intersection of $d\alpha$, the points l, β, γ will necessarily be collinear; since, of the nine points in which the three lines $ab\gamma$, $ac\beta$, and ald intersect the cubic, six are situated on the two lines aad and $bc\alpha$. (Salmon's *Higher Plane Curves*, Art. 133.) Hence, to solve the problem, when the three points α, β, γ are given, it is merely necessary, first to join $\beta\gamma$, which will cut the cubic in l, next to join $l\alpha$, which will intersect again in d, and finally to draw any tangent from d to the cubic. The point of contact a of this tangent will then be one corner of the required triangle; $a\gamma$ and $a\beta$ will be two of its sides, and they will intersect the cubic in the remaining corners b and c, respectively. When the curve is of the sixth class, there are obviously four solutions.

1616. (Proposed by GEOMETRICUS.)—Let O_1, O_2, O_3 be the centres of the escribed circles touching the sides BC, CA, AB respectively of the triangle ABC; and K_1, K_2, K_3 the middle points of these sides; prove that O_1K_1, O_2K_2, O_3K_3 are concurrent.

I. *Solution by* W. Hopps; D. M. Anderson; *and others.*



It is obvious, from the Solution of Prop. 73, McDowell's *Exercises*, that the triangle ABC is inscribed in, and co-polar to, the triangle formed by joining each pair of the points O_1, O_2, O_3; hence (*ibid.* Prop. 183), these triangles are co-axial, that is to say, the intersections (P, Q, R) of each pair of their corresponding sides are collinear. Moreover, as the lines passing through each pair of the points K_1, K_2, K_3 obviously bisect the three diagonals (PB, CQ, AR) of the complete quadrilateral PAQCR in p, q, r, and as these points are collinear (*ibid.* Prop. 22), it is manifest that the triangles $K_1K_2K_3$ and $O_1O_2O_3$ are co-axial, and consequently they are co-polar (*ibid.* Prop. 183), or, in other words, the lines O_1K_1, O_2K_2, O_3K_3 are concurrent.

Corollary.—From what precedes it is plain that this question is but a particular case of the following

Theorem.—If a triangle inscribed in a given triangle is co-polar thereto, the triangle formed by joining each pair of the middle points of the sides of the inscribed triangle is also co-polar to the given triangle.

II. *Solution by* Archer Stanley.

From the equality of BK_1 and CK_1, it follows at once that

$$\frac{\sin BO_1K_1}{\sin CO_1K_1} = \frac{CO_1}{BO_1},$$

and for a similar reason, that

$$\frac{\sin CO_2K_2}{\sin AO_2K_2} = \frac{AO_2}{CO_2}, \quad \text{and} \quad \frac{\sin AO_3K_3}{\sin BO_3K_3} = \frac{BO_3}{AO_3}.$$

But the triangle $O_1O_2O_3$ is circumscribed to ABC, and through the centre of the circle inscribed to the latter pass the three connectors O_1A, O_2B, O_3C; hence, by the theorem of Ceva, the product of the above three ratios of segments is equal to -1, and consequently also

$$\frac{\sin O_3O_1K_1 \,.\, \sin O_1O_2K_2 \,.\, \sin O_2O_3K_3}{\sin O_2O_1K_1 \,.\, \sin O_3O_2K_2 \,.\, \sin O_1O_3K_3} = -1.$$

This virtually reduces the question to the converse of Ceva's theorem, which is known to be true; for the above sines may obviously be replaced by the six segments of the sides of the triangle $O_1O_2O_3$, which the several angles subtend.

[Note.—Since O_2B, O_3C are respectively perpendicular to O_1O_3, O_1O_2, it is easy to show that O_1K_1 will divide O_2O_3 into parts which have to one another the duplicate ratio of the adjacent sides of the triangle $O_1O_2O_3$; and the like may be proved of O_2K_2, O_3K_3; whence it follows at once, by Ceva's theorem, that O_1K_1, O_2K_2, O_3K_3 are concurrent; their point of intersection being, in fact, such that the sum of the squares on the perpendiculars drawn

therefrom on the sides of the triangle $O_1O_2O_3$ is a minimum, and these perpendiculars are, moreover, proportional to the sides on which they fall.

Mr. FITZGERALD and Mr. BILLS prove the theorem by trilinear coordinates. Taking ABC as triangle of reference, the equations of O_1K_1, O_2K_2, O_3K_3 are readily found to be as follows, viz.,

$$b(\alpha+\beta) = c(\alpha+\gamma), \quad a(\alpha+\beta) = c(\beta+\gamma), \quad a(\alpha+\gamma) = b(\beta+\gamma);$$

which are all satisfied by the relations

$$\frac{\alpha}{s_1} = \frac{\beta}{s_2} = \frac{\gamma}{s_3} = \frac{2\Delta}{as_1+bs_2+cs_3}, \text{ (where } s_1 = s-a, \text{ \&c.)},$$

thus showing that O_1K_1, O_2K_2, O_3K_3 meet in this point.—EDITOR.]

1619. (Proposed by the Rev. J. BLISSARD.)—Prove that

$$(1)\ldots\ldots x\sin\theta - \tfrac{1}{2}x^2\sin 2\theta + \tfrac{1}{3}x^3\sin 3\theta - \&c. = \tan^{-1}\left(\frac{x\sin\theta}{1+x\cos\theta}\right);$$

$$(2)\ldots\ldots x\sin\theta - \tfrac{1}{3}x^3\sin 3\theta + \tfrac{1}{5}x^5\sin 5\theta - \&c. = \tfrac{1}{4}\log\left(\frac{1+2x\sin\theta+x^2}{1-2x\sin\theta+x^2}\right).$$

Solution by W. A. WHITWORTH; *and* E. FITZGERALD.

(1.) Let $u = x\cos\theta - \dfrac{x^2}{2}\cos 2\theta + \dfrac{x^3}{3}\cos 3\theta - \&c.,$

and $v = x\sin\theta - \dfrac{x^2}{2}\sin 2\theta + \dfrac{x^3}{3}\sin 3\theta - \&c.$

then
$$u+iv = x(\cos\theta + i\sin\theta) - \frac{x^2}{2}(\cos\theta+i\sin\theta)^2 + \&c.$$
$$= \log(1+x\cos\theta+ix\sin\theta)$$
$$= \tfrac{1}{2}\log(1+2x\cos\theta+x^2) + i\tan^{-1}\left(\frac{x\sin\theta}{1+x\cos\theta}\right).$$

Hence, equating rational and irrational parts, we have

$$u = \tfrac{1}{2}\log(1+2x\cos\theta+x^2) \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

$$v = \tan^{-1}\left(\frac{x\sin\theta}{1+x\cos\theta}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}.$$

(2.) Let $u = x\cos\theta - \dfrac{x^3}{3}\cos 3\theta + \dfrac{x^5}{5}\cos 5\theta - \&c.$

and $v = x\sin\theta - \dfrac{x^3}{3}\sin 3\theta + \dfrac{x^5}{5}\sin 5\theta - \&c.;$

then $u+iv = x(\cos\theta+i\sin\theta)-\frac{x^3}{3}(\cos\theta+i\sin\theta)^3+\&c.$

$$= \tan^{-1}(x\cos\theta+ix\sin\theta) = \frac{1}{2i}\log\frac{1-x\sin\theta+ix\cos\theta}{1+x\sin\theta-ix\cos\theta}$$

$$= \frac{1}{2i}\left\{\tfrac{1}{2}\log(1-2x\sin\theta+x^2)+i\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right)\right.$$

$$\left.-\tfrac{1}{2}\log(1+2x\sin\theta+x^2)+i\tan^{-1}\left(\frac{x\cos\theta}{1+x\sin\theta}\right)\right\}.$$

Hence, equating rational and irrational parts, we have

$$u = \tfrac{1}{2}\left\{\tan^{-1}\frac{x\cos\theta}{1-x\sin\theta}+\tan^{-1}\frac{x\cos\theta}{1+x\sin\theta}\right\} = \tfrac{1}{2}\tan^{-1}\left(\frac{2x\cos\theta}{1-x^2}\right)\ldots\ldots(C),$$

$$v = \tfrac{1}{4}\log\left(\frac{1+2x\sin\theta+x^2}{1-2x\sin\theta+x^2}\right)\ldots\ldots\ldots\ldots\ldots\ldots(D).$$

(B) and (D) are Mr. Blissard's results, but (A) and (B) appear to be equally noteworthy.

1597. (Proposed by C. Taylor, M.A.)—If SY, HZ be focal perpendiculars on the tangent at P to an ellipse, and SY′, HZ′ perpendiculars on the tangents from P to a confocal ellipse; prove that the rectangle YY′. ZZ′ is equal to the difference of the squares on the semi-axes.

Solution by R. Tucker, M.A.

Draw the normal PG at P, and denote the angle Y′PZ′ between the tangents to the confocal conic by θ; then, since a circle can be drawn round SPYY′, we have

$$YY' = SP\sin YPY' = \rho\cos\tfrac{1}{2}\theta; \quad \text{and similarly} \quad ZZ' = \rho'\cos\tfrac{1}{2}\theta;$$

where ρ, ρ' stand for SP, HP respectively;

hence, $$YY'.ZZ' = \rho\rho'\cos^2\tfrac{1}{2}\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i.})$$

Now $2\cos^2\frac{1}{2}\theta = 1+\cos\theta = 1+\frac{\rho^2+\rho'^2-4a'^2}{2\rho\rho'}$ (Salmon's *Conics*, p. 198)

$$= \frac{(\rho+\rho')^2-4a'^2}{2\rho\rho'} = \frac{4(a^2-a'^2)}{2\rho\rho'};$$

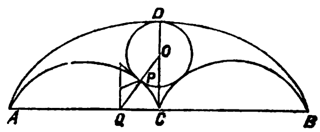
hence, by (i.), $$YY'.ZZ' = a^2-a'^2 = b^2-b'^2,$$

which proves the theorem.

1574. (Proposed by H. J. Purkiss, B.A.)—A straight line AB is bisected in C, and upon the same side of AB, AC, CB cycloids are described;

a circle is then drawn touching the three cycloids. Show that, if θ be the angle which the radius of the circle drawn to the point of contact of the cycloid on AC makes with AC, then $(1+\sin\theta)\,\theta = (2+\sin\theta)\cos\theta$.

Solution by the PROPOSER.



Let CD be the axis of the large cycloid; the centre O of the circle will evidently lie on it. Let P be the point of contact with the cycloid on AC, and let OP produced meet AC in Q; then OQC $=\theta$. Let a be the radius of the generating circle of either of the small cycloids, and let OP $=r$. Then, since QC $=2a\theta$, we have

$$2a\theta\tan\theta = \mathrm{CO} = \mathrm{CD}-\mathrm{OD} = 4a-r,$$

$$2a\theta\sec\theta = \mathrm{OQ} = \mathrm{PQ}+\mathrm{OP} = 2a\sin\theta+r.$$

Adding, dividing by $2a$, and multiplying by $\cos\theta$, we have the required result.

1614. (Proposed by J. GRIFFITHS, M.A.)—One focus of a conic, self conjugate with respect to a given triangle, moves on a straight line; find the locus of the other focus.

Solution by the PROPOSER.

If the given triangle be taken as the triangle of reference, the trilinear equation of a conic self-conjugate with respect to it will be

$$l\alpha^2+m\beta^2+n\gamma^2=0.$$

Also the equations of the two tangents to the curve which are parallel to the side $\alpha=0$, for instance, will be of the forms

$$pa\alpha+b\beta+c\gamma=0,\quad -pa\alpha+b\beta+c\gamma=0.$$

Hence, if λ denote the semi-minor axis of the conic; and (α',β',γ'), (α,β,γ) be the foci, we have, by a well known theorem,

$$(pa\alpha'+b\beta'+c\gamma')\,(pa\alpha+b\beta+c\gamma)=\lambda^2(p-1)^2a^2,$$

$$(-pa\alpha'+b\beta'+c\gamma')\,(-pa\alpha+b\beta+c\gamma)=\lambda^2(p+1)^2a^2.$$

Hence, by subtraction, we get

$$2pa\alpha\,(b\beta'+c\gamma')+2pa\alpha'\,(b\beta+c\gamma)=-4\lambda^2pa^2,$$

whence $\alpha'=\dfrac{\lambda^2a+\Delta\alpha}{a\alpha-\Delta}$; similarly $\beta'=\dfrac{\lambda^2\beta+\Delta\beta}{b\beta-\Delta}$; and $\gamma'=\dfrac{\lambda^2c+\Delta\gamma}{c\gamma-\Delta}$;

where Δ is the area of the triangle of reference.

It remains now to eliminate α', β', γ', and λ^2. This is easily done by means of the equations just found, and the two following ones,

$$a\alpha' + b\beta' + c\gamma' = 2\Delta, \quad A\alpha' + B\beta' + C\gamma' = 0,$$

($A\alpha + B\beta + C\gamma = 0$ being the equation of the given straight line.)

The resulting equation is

$$\left(\frac{a^2}{\Delta - a\alpha} + \frac{b^2}{\Delta - b\beta} + \frac{c^2}{\Delta - c\gamma}\right)\left(\frac{A\alpha}{\Delta - a\alpha} + \frac{B\beta}{\Delta - b\beta} + \frac{C\gamma}{\Delta - c\gamma}\right)$$
$$= \left(\frac{Aa}{\Delta - a\alpha} + \frac{Bb}{\Delta - b\beta} + \frac{Cc}{\Delta - c\gamma}\right)\left(2 + \frac{a\alpha}{\Delta - a\alpha} + \frac{b\beta}{\Delta - b\beta} + \frac{c\gamma}{\Delta - c\gamma}\right),$$

which reduces to

$$\left(\frac{b}{B} - \frac{c}{C}\right)\left(\frac{\beta}{b} - \frac{\gamma}{c}\right)\frac{-a\alpha + b\beta + c\gamma}{Aa} + \left(\frac{c}{C} - \frac{a}{A}\right)\left(\frac{\gamma}{c} - \frac{\alpha}{a}\right)\frac{a\alpha - b\beta + c\gamma}{Bb}$$
$$+ \left(\frac{a}{A} - \frac{b}{B}\right)\left(\frac{\alpha}{a} - \frac{\beta}{b}\right)\frac{a\alpha + b\beta - c\gamma}{Cc} + \left\{\frac{(a\alpha - b\beta + c\gamma)(a\alpha + b\beta - c\gamma)}{bc\,.\,BC}\right.$$
$$\left. + \frac{(-a\alpha + b\beta + c\gamma)(a\alpha + b\beta - c\gamma)}{ca\,.\,CA} + \frac{(-a\alpha + b\beta + c\gamma)(a\alpha - b\beta + c\gamma)}{ab\,.\,AB}\right\} = 0,$$

the equation of the locus required.

If $A : B : C = a : b : c$, that is to say, if the curve is a parabola, the above conic-locus becomes the nine-point circle of the triangle of reference, as we know ought to be the case.

Note on the Proof ordinarily given of the divergency of the Harmonic Series.

By I. Todhunter, M.A., F.R.S.

The common proof of the divergency of the series

$$\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{3} + \ldots\ldots\ldots$$

seems to be quite sound, notwithstanding the objections urged in the last Number of the *Educational Times*. (See p. 13 of this Vol. of the *Reprint*.) For a series is divergent when, by taking terms enough, we get a sum greater than *any assigned number*. Now suppose n to be the number assigned, then we get a sum greater than n by taking $2^{2n} - 1$ terms of the series.

We do not assert that the sum of a definite number of terms of the proposed series is greater than the sum of the *same* number of terms of the series

$$\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \ldots\ldots\ldots\ldots$$

and thus the objection drawn from the comparison of $\log.\dfrac{1}{1-x}$ with $\dfrac{1}{1-x}$ is obviated.

1468. (Proposed by W. K. CLIFFORD.)—Given the centre of a conic, and a conjugate triad; to construct for the directions of the asymptotes.

I. *Solution by the* REV. R. TOWNSEND, M.A.; *and the* PROPOSER.

If O be the given centre of the conic; A, B, C the three points of the self-conjugate triad; and OX, OY, OZ the three lines through O parallel to BC, CA, AB respectively; the two double rays (real or imaginary) OM and ON of the involution determined by the three angles AOX, BOY, COZ are the two asymptotes required. For the three pairs of conjugates, OA and OX, OB and OY, OC and OZ, determining the involution, being evidently pairs of conjugate diameters of the conic, divide, therefore, harmonically the angle (real or imaginary) MON determined by the two asymptotes OM and ON.

The same construction (with some slight and obvious modifications) applies also to the following more general problem, of which the above is evidently a particular case: viz., Given a point and a line, pole and polar with respect to a conic, and a conjugate triad; to construct the two tangents (real or imaginary) from the point to the curve, and the two intersections (real or imaginary) of the line with the curve. For if P and L be the point and line; A, B, C, as before, the three points of the triad; X, Y, Z and X′, Y′, Z′ the six intersections of L with BC, CA, AB, and with PA, PB, PC, respectively; then, as X and X′, Y and Y′, Z and Z′ are evidently pairs of conjugate points with respect to the conic, the two double points M and N of the involution determined by the three segments XX′, YY′, ZZ′, as cutting them all harmonically, are the two intersections required; and, as PX and PX′, PY and PY′, PZ and PZ′, are evidently pairs of conjugate lines with respect to the conic, the two double rays PM and PN of the involution determined by the three angles XPX′, YPY′, ZPZ′, as cutting them all harmonically, are the two tangents required.

The two corresponding problems in Geometry of three dimensions—viz., Given, of a quadric, the centre and a self-conjugate tetrahedron, to construct the asymptotic cone of the surface; or, more generally: Given, of a quadric, a point and plane, pole and polar to each other, and a self-conjugate tetrahedron, to construct the tangent cone from the point to the surface, and the conic of intersection of the plane with the surface—may be readily solved by application of the above.

COROLLARY I. Let any two straight lines parallel to two conjugate diameters be called *conjugate* with respect to a conic; then it is shown above that the pairs of lines 12, 34; 13, 24; 14. 23, joining the points 1234, are conjugates with respect to the conic which has the point 1 for a centre, and 234 for a conjugate triad. But the symmetry of this statement shows that they are also conjugates with respect to the conic which has any other of the four points for centre, and the remaining three for a conjugate triad. We may draw four such conics; and since the asymptotes are determined in direction by *two* pairs of conjugates, it follows that these four conics are all similar and similarly situated. So, in the more general case, we shall have four conics intersecting in two points on the given straight line.

COROLLARY II. Let a straight line and plane, drawn parallel to any diameter and its conjugate diametral plane, be called *conjugates* with respect to a conicoid. Then if we are given five points 12345, of which 1 is the centre, and 2345 a self-conjugate tetrahedron of a given conicoid, it is evident that since 2 is the pole of the plane 345, 12 is conjugate to 345, and so on. We thus get four pairs of conjugates. Again, since 23 is the polar line of 45,

123 is conjugate to 45, and 145 to 23, and so on. This gives us six more pairs of conjugates. But this amounts to saying that if we join any three of the five points by a plane, and the other two by a line, the line and plane are conjugates. This statement makes no mention of the particular point taken for centre; and we conclude as before, that if five conicoids are drawn, by taking each of five points in succession for centre, and the remaining four for a self-conjugate tetrahedron, these five conicoids will be similar and similarly situated. A line and its conjugate plane cut the plane at infinity in a point and line which are pole and polar with respect to the section which the plane at infinity makes of the conicoid. The problem is therefore equivalent to that of describing a conic, being given the poles of certain lines. Three points and their polars are sufficient to determine a conic; for let A, B, C be the points, and let AB meet the polars of A and B in P, Q respectively. Then the foci of the involution determined by AP, BQ, are evidently points on the conic. In this way we can determine six points on the sides of the triangle ABC, and six more on the sides of the reciprocal triangle; and it would be interesting to prove *a priori* that these twelve points must lie on the same conic, when the triangles are in perspective.

Corollary III. In the plane case we are given three pairs of conjugates to determine two points at infinity; and we conclude that any transversal is cut in involution by the six lines joining four points. Similarly we conclude from the solid problem that "if five points in space are joined every way by ten lines and ten planes, the system will be cut by any plane in a system of points and lines which are poles and polars with respect to a certain conic." The analogy of this relation of points and lines with involution may be illustrated analytically. Let $U \equiv ax^2 + by^2 = 0$ be a pair of points; then if we put Δ for $\left(\xi\frac{d}{dx} + \eta\frac{d}{dy}\right)$, a point and its harmonic conjugate will be represented by the equations $\begin{vmatrix} x, & y \\ \xi, & \eta \end{vmatrix} = 0$, and $\Delta U = 0$. And a system of such harmonic conjugates is of course a system in involution. Next let U represent a conic, and Δ stand for $\left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)$, then a point and its polar will be represented by the equations

$$\left\|\begin{matrix} x, & y, & z \\ \xi, & \eta, & \zeta \end{matrix}\right\| = 0, \text{ and } \Delta U = 0,$$

and the analogy is obvious.

Corollary IV. Lastly, the four conics mentioned in Cor. I. are all similar to the *nine-point conic* of the quadrangle, or locus of centres of all conics through the four points. This proposition was set in a problem paper, at St. John's College, Cambridge, in Dec. 1862; but I do not know to whom it is due. It follows at once from the equation to the nine-point conic given in Art. 2. of the Solution to Quest. 1443; for an equation of the second degree in x, y, 1, in which the coefficient of xy is zero, obviously represents a conic with respect to which the axes are conjugates. Thus the lines 12, 34; 13, 24; 14, 23, are conjugates with respect to the nine-point conic, and therefore its asymptotes are parallel to those of the other four. These, therefore, are ellipses when the quadrangle is re-entrant, and hyperbolas when it is convex.

II. *Solution by* ARCHER STANLEY.

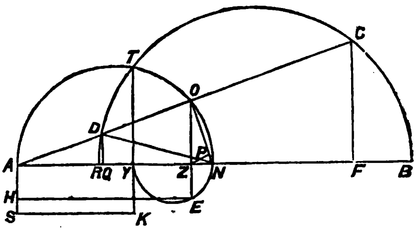
Connect the given centre O of the conic with any two corners A and B of the given triangle, and let a and b be the points in which the connecting lines cut any circle through O. Let this circle be next cut in α and β by parallels through O to the sides opposite A and B respectively. Draw the chords $a\alpha$ and $b\beta$, and from their intersection P draw two lines touching the circle in p and p', then Op and Op' will be the required asymptotes.

This construction, as will be at once recognized, gives the double rays of the involution determined by the conjugate rays Oa, Oα, and Ob, Oβ. Now the polar, with respect to the conic, of the infinitely distant point of BC being obviously OA, Oa and Oα are conjugate diameters, as also are Ob and Oβ. But, as is well known, the conjugate diameters of a conic form a pencil in involution, the double rays of which are precisely the required asymptotes.

The same construction also gives the tangents, in O, to the two parabolas which can be inscribed to the given trilateral so as to pass through O. For the tangents from O to all the conics (parabolas) inscribed to the quadrilateral which consists of the trilateral and the line at infinity, form a pencil in involution, of which Oa, Oα ; Ob, Oβ, (and similarly Oc, Oγ) are pairs of conjugate rays; and of these parabolas the two which pass through O are obviously touched in that point by the double rays of this involution.

1596. (Proposed by J. O'CALLAGHAN.)—From a given point in the diameter (produced) of a given semicircle to draw a line, cutting the circumference in two points, from which if perpendiculars be drawn to the diameter, the trapezoid thus formed may be given or a maximum.

Solution by ALPHA; *the* PROPOSER; E. FITZGERALD; *and others.*



Let RCB be the given semicircle, N its centre, and A the given point in the diameter BR produced. On AN draw the semicircle ATON, cutting RCB in T; from T draw TY perpendicular to AB, and produce TY to K, making the rectangle AYKS equal to *half* the given area. On NY draw the semicircle NEY; through K, along AB, AS as asymptotes, draw a rectangular hyperbola cutting the semicircle NEY in E; and from E draw EZO perpendicular to AB, meeting the semicircle ATN in O: then ADOC will be a line drawn so that the trapezoid CDQF is equal to the given area.

For draw ZP perpendicular to ON; then we have

$$AN \cdot YZ = AN \cdot NY - AN \cdot NZ = NT^2 - NO^2 = ND^2 - NO^2 = DO^2;$$

$$DO : QZ = ZO : OP, \text{ or } DO \cdot OP = QZ \cdot ZO;$$

$$AN : AZ = ON : OP, \text{ or } AN \cdot OP = AZ \cdot ON;$$

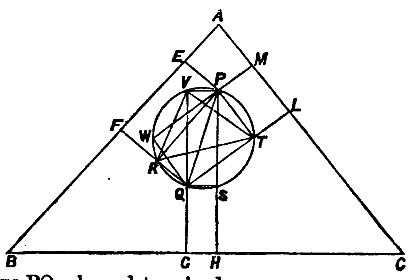
$$\therefore\ AZ^2 . ZE^2 . AN = AZ^2 . YZ . ZN . AN = AZ^2 . ON^2 . YZ$$
$$= AN^2 . OP^2 . YZ = AN . DO^2 . OP^2;$$
$$\therefore\ AZ . ZE = DO . OP = QZ . ZO.$$

Now QZ . ZO is equal to half the area of the trapezoid CDQF; and, by a property of the hyperbola, AZ . ZE = AY . YZ = half the given area, by construction; therefore the line ADOC has been drawn so that the trapezoid CDQF is equal to the given area.

There are in general *two* positions of the line ADOC which satisfy the conditions of the problem, corresponding to the two positions of E, that is, to the two intersections of the hyperbola with the semicircle; and when these two positions of E coincide, that is to say, when the hyperbola just *touches* the semicircle, the rectangle AZ . ZE, and therefore the trapezoid CDQF, will be a *maximum;* and in this case, by a well-known property of the hyperbola, the portion of the common tangent to the semicircle and the hyperbola, intercepted between AB and AS, will be *bisected* at the point E.

1575. (Proposed by Mr. A. Renshaw.)—A triangle ABC and a point P being given; find the locus of another point Q, such that if perpendiculars be drawn from Q on the sides BC, CA, AB, cutting the circle on PQ in the points V, T, R, the perimeter of the triangle VTR shall be constant.

Solution by the Proposer; *and others.*



Join RP, PT, TQ; then $\angle$ RPT = A, VQR = B, VQT = C; hence we have TR + RV + VT = PQ (sin A + sin B + sin C) = k (a *constant*) by the question;

$$\therefore\ PQ = \frac{k}{\sin A + \sin B + \sin C}$$

[= a *constant.*]

Therefore the locus of Q is a circle drawn round P as centre, with the constant radius PQ, above determined.

Corollary. Let the trilinear coordinates of P be (α, β, γ), and of Q, $(\alpha', \beta', \gamma')$; then

$$\sin^2 A . PQ^2 = TR^2 = (\gamma-\gamma')^2 + (\beta-\beta')^2 + 2(\gamma-\gamma')(\beta-\beta')\cos A,$$
$$\sin^2 B . PQ^2 = RV^2 = (\alpha-\alpha')^2 + (\gamma-\gamma')^2 + 2(\gamma-\gamma')(\alpha-\alpha')\cos B,$$
$$\sin^2 C . PQ^2 = VT^2 = (\beta-\beta')^2 + (\alpha-\alpha')^2 + 2(\alpha-\alpha')(\beta-\beta')\cos C,$$

$$\therefore\ PQ^2 (=\delta^2) = \frac{2\left\{\begin{array}{l}(\alpha-\alpha')^2 + (\beta-\beta')^2 + (\gamma-\gamma')^2 + (\gamma-\gamma')(\beta-\beta')\cos A\\ \quad + (\gamma-\gamma')(\alpha-\alpha')\cos B + (\alpha-\alpha')(\beta-\beta')\cos C\end{array}\right\}}{\sin^2 A + \sin^2 B + \sin^2 C}.$$

1563. (Proposed by H. J. PURKISS, B.A.)—A body is referred to principal axes, the moments of inertia about which are A, B, C respectively; show that the sum of the moments of inertia about any pair of rectangular axes through the origin in the plane $lx+my+nz=0$ is

$$(B+C)\,l^2+(C+A)\,m^2+(A+B)\,n^2.$$

Solution by the PROPOSER.

The moment of inertia of the body about any axis through the origin is equal to the reciprocal of the square of the radius-vector of the ellipsoid

$$Ax^2+By^2+Cz^2=1$$

in the same direction. Now the reciprocals of the semi-axes of the section made by the plane

$$lx+my+nz=0$$

are given by the equation

$$\frac{l^2}{u^2-A}+\frac{m^2}{u^2-B}+\frac{n^2}{u^2-C}=0;$$

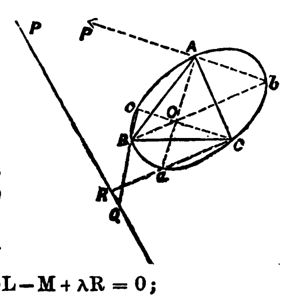
hence the sum of the two values of u^2 is

$$(B+C)\,l^2+(C+A)\,m^2+(A+B)\,n^2;$$

therefore, since the sum of the reciprocals of the squares of any pair of perpendicular radii of an ellipse is constant, the above expression gives the value of the sum of the moments of inertia of the body about any pair of rectangular axes through the origin in the plane in question.

1623. (Proposed by F. D. THOMSON, M.A.)—ABC is a triangle inscribed in a conic; Aa, Bb, Cc are chords drawn through the point O, of which the polar is PQR; Ab, Bc, Ca meet PQR in P, Q, R, respectively. Show that if S be any point on the conic, SP, SQ, SR meet the sides of the triangle ABC in points which lie in a straight line. Deduce the corresponding theorem for the circle.

Solution by the PROPOSER.



Let $R=0$ be the equation to PQR; then the equation to the conic is of the form

$$LM=R^2\ldots\ldots\ldots\ldots\ldots\text{(i.)}$$

Let the equation to the chord AB be

$$abL-(a+b)\,R+M=0\ldots\ldots\ldots\text{(ii.)},$$

and similar equations for BC, CA. Then the equation to Ab will be, writing $-b$ for b,

$$abL+(a-b)\,R-M=0\ldots\ldots\ldots\ldots\text{(iii.)}$$

Let the point S be determined by ($\mu L=R$, $\mu R=M$); then the equation to SP is of each of the forms

$$\kappa(\mu L-R)+\mu R-M=0,\quad abL-M+\lambda R=0;$$

therefore its equation is

$$abL+\frac{\mu^2-ab}{\mu}R-M=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv.)}$$

This meets (ii.) at the point given by

$$\frac{L}{-(a+b)+\dfrac{\mu^2-ab}{\mu}}=\frac{R}{-2ab}=\frac{M}{-ab\left\{(a+b)+\dfrac{\mu^2-ab}{\mu}\right\}} \;\ldots\ldots\ldots\; (\gamma).$$

Similarly for the point (α). Hence the coefficients of L, R, M, in the equation to $\gamma\alpha$ will be proportional to

$$2ab^2c\begin{vmatrix}1, & a+b+\mu-\dfrac{ab}{\mu}\\ 1, & b+c+\mu-\dfrac{bc}{\mu}\end{vmatrix};\quad b\begin{vmatrix}a\left\{a+b+\mu-\dfrac{ab}{\mu}\right\}, & a+b-\mu+\dfrac{ab}{\mu}\\ c\left\{b+c+\mu-\dfrac{bc}{\mu}\right\}, & b+c-\mu+\dfrac{bc}{\mu}\end{vmatrix};$$

$$2b\begin{vmatrix}a+b-\mu+\dfrac{ab}{\mu}, & a\\ b+c-\mu+\dfrac{bc}{\mu}, & c\end{vmatrix}.$$

The equation of $\gamma\alpha$ is hence found to be

$$2\mu abc\,L+\left\{\mu^3+\mu^2(a+b+c)-\mu(ab+bc+ca)-abc\right\}R-2\mu^2M=0.$$

This is symmetrical with respect to a, b, c; hence the points α, β, γ are in a straight line.

COROLLARY.—If the figure be projected so that PQR may become the line at infinity, and the conic a circle, O will become the centre of the circle, and SP, SQ, SR lines perpendicular to the sides of the triangle ABC. Hence "the three perpendiculars drawn from a point in the circumference of a circle upon the sides of an inscribed triangle have their feet in a straight line."

1565. (Proposed by C. TAYLOR, M.A.)—Prove *geometrically* that

$$\sin(\theta-\phi)\sin(\theta+\phi)=\sin^2\theta-\sin^2\phi.$$

I. *Solution by the* PROPOSER.

Let PQ, QR be equal arcs of a circle; AB a diameter; O the intersection of PR, AQ. Then (since $\angle$QPR=QAR=QAP, &c.) the triangles QPO, QAP are similar, and AQ . QO = QP². Also AQ . AO = AP . AR, by similar triangles AOR, APQ. Hence, by addition, we have AQ² = QP² + AP . AR, or AP . AR = AQ² − QP².

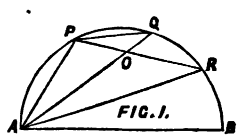


FIG. I.

(i.) Let $\angle ABQ=\theta$; $\angle PBQ=\phi=RBQ$.
Then $\sin(\theta-\phi)\sin(\theta+\phi)=\sin^2\theta-\sin^2\phi$.

(ii.) Let $\angle BAQ=\theta$; $\angle RAQ=\phi=PAQ$.
Then $\cos(\theta+\phi)\cos(\theta-\phi)=\cos^2\theta-\sin^2\phi$.

II. *Solution by* ALPHA; H. MURPHY; *and others.*

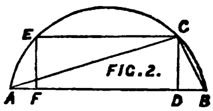


FIG. 2.

In a circle (Fig. 2) whose radius is unity, let the arc $AC=2\theta$, and the arc $CB=AE=2\phi$.

Draw CD, EF perpendicular to AB.

Then $AC^2-CB^2=AD^2-DB^2=AB\,.\,DF$;
or, since the chord of an arc is twice the sine of half the arc,

$$4\sin^2\theta-4\sin^2\phi=2\sin(\theta+\phi)\,.\,2\sin(\theta-\phi),$$

therefore $$\sin^2\theta-\sin^2\phi=\sin(\theta+\phi)\sin(\theta-\phi).$$

[NOTE.—The property in the Question may be enunciated, geometrically, in the following form, which is analogous to that of Euclid II. 5.

If a circular arc be divided into two equal and also into two unequal parts, the rectangle contained by the chords of the two unequal arcs, together with the square on the chord of the arc between the points of section, will be equal to the square on the chord of half the arc.—EDITOR.]

1518. (Proposed by STEPHEN FENWICK, F.R.A.S.)—A ball of weight w is projected up a smooth plane inclined at an angle α to the horizon from a given point in the plane, with a velocity β. The resistance of the air being taken to vary as the velocity of the ball, or as kv, find the position of the ball at the instant it has attained a velocity β in its descent; and thence show that the point thus determined is below the point of projection.

Solution by the PROPOSER; F. D. THOMSON, M.A.; *and others.*

The retarding pressure to the motion up the plane being $w\sin\alpha+kv$, the retarding force is

$$-\frac{dv}{dt}=\frac{w\sin\alpha+kv}{w}\,g,$$

or if we put $k=\dfrac{w}{\mu}$, and change signs,

$$\frac{dv}{dt}=-\frac{g}{\mu}(\mu\sin\alpha+v).$$

But if x be the space described in a time t,

$$v=\frac{dx}{dt}=\frac{dx}{dv}\cdot\frac{dv}{dt}=-\frac{g}{\mu}(\mu\sin\alpha+v)\frac{dx}{dv},$$

or, $$\frac{dx}{dv}=-\frac{\mu}{g}\,\frac{v}{\mu\sin\alpha+v}=-\frac{\mu}{g}+\frac{\mu^2\sin\alpha}{g}\,\frac{1}{\mu\sin\alpha+v}.$$

Integrating the last equation, and determining the constant of integration by the condition that, when $x=0$, $v=\beta$, we get

$$x = \frac{\mu}{g}(\beta-v) + \frac{\mu^2 \sin \alpha}{g} \cdot \log \frac{\mu \sin \alpha + v}{\mu \sin \alpha + \beta}.$$

Making $v=0$, the space to the highest point is hence

$$x_1 = \frac{\mu\beta}{g} + \frac{\mu^2 \sin \alpha}{g} \log \frac{\mu \sin \alpha}{\mu \sin \alpha + \beta}.$$

When the ball attains a velocity β in its descent, then $v=-\beta$, and the distance from the point of projection is therefore

$$x_2 = \frac{2\mu\beta}{g} + \frac{\mu^2 \sin \alpha}{g} \log \frac{\mu \sin \alpha - \beta}{\mu \sin \alpha + \beta}.$$

This gives, on developing the last term,

$$x_2 = \frac{2\mu\beta}{g} - \frac{2\mu\beta}{g} - \frac{2}{3}\,\frac{\beta^3}{\mu g \sin^2 \alpha} - \&c. = -\frac{2}{3}\,\frac{\beta^3}{\mu g \sin^2 \alpha} - \&c,$$

a *negative* quantity, showing that the ball is *below* the point of projection.

If the resistance of the air is neglected, then $k=0$, or $\mu=\infty$, and therefore $x_2=0$, as it ought to be.

1550. (Proposed by JOHN CASEY, B.A.)—To show by a simple geometrical proof how to represent the amplitude of elliptic functions of the first order, and illustrate it independently of analysis by motion in a vertical circle; and to show also geometrically, when the velocity of projection is that due to a fall from the highest point, that the time of reaching the highest point in the ascent will be infinite.

Solution by the PROPOSER.

CHASLES occupies a whole chapter of his *Géométrie Supérieure* in showing that the amplitude of elliptic functions may be represented by coaxial circles. Sometime since, while reading that part of his work, the following simple method of doing the same occurred to me.

1. Let X, Y, be two circles whose centres are C, C′; P one of their limiting points; and RA their radical axis. Then, if ab, $a'b'$ be two consecutive positions of a tangent of Y, which is a chord of X, we have, since aa' and bb' are tangents to a circle coaxial with XY (*Townsend's Modern Geometry*, Vol. I., Art. 194),

$$\frac{aa'}{aP + a'P} = \frac{bb'}{bP + b'P},$$

and therefore, in the limit,

$$\frac{aa'}{a'P} = \frac{bb'}{b'P}.$$

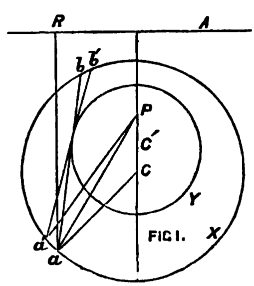


FIG I.

Now let $\angle \mathrm{CP}a = \theta$, $a\mathrm{P}a' = \delta\theta$, and $\dfrac{\mathrm{CP}}{\mathrm{C}a} = c$; then we have

$$\frac{aa'}{a'\mathrm{P}} = \frac{\delta\theta}{\sin \mathrm{P}aa'} = \frac{\delta\theta}{\sqrt{(1-\sin^2 \mathrm{C}a\mathrm{P})}} = \frac{\delta\theta}{\sqrt{(1-c^2\sin^2\theta)}}.$$

Hence it is evident that the integral $\displaystyle\int_{\mathrm{CP}a}^{\mathrm{CP}b} \frac{d\theta}{\sqrt{(1-c^2\sin^2\theta)}}$ is *constant*, whatever be the position of ab.

From the foregoing construction it is easy to infer the usual elementary properties of Elliptic integrals. (See Hymers' *Integral Calculus.*)

2. If a particle be projected from any point a in X, with the velocity due to a fall from the radical axis, we have

$$v = \frac{aa'}{dt}, \quad \text{also} \quad v = \sqrt{(2g \,.\, a\mathrm{R})} = \sqrt{\left(2g \,.\, \frac{a\mathrm{P}^2}{2\mathrm{CP}}\right)} = \sqrt{\left(\frac{g}{\mathrm{CP}}\right)} \,.\, a\mathrm{P};$$

therefore $$dt = \sqrt{\left(\frac{\mathrm{CP}}{g}\right)} \,.\, \frac{aa'}{a\mathrm{P}} = \sqrt{\left(\frac{\mathrm{CP}}{g}\right)} \,.\, \frac{d\theta}{\sqrt{(1-c^2\sin^2\theta)}}.$$

Hence the time of describing any portion of a vertical circle depends on an elliptic integral.

3. If X, Y be two circles touching at the highest point, then their radical axis is a common tangent at that point. From the lowest point of X draw a succession of chords touching Y (as in Fig. 2); it is evident there will be an infinite number before we reach the highest point. But if a particle be projected from a with the velocity due to a fall from the highest point, it is evident from Arts. 1 and 2 (see also the *Messenger of Mathematics*, vol. i., p. 54) that the times of describing the arcs ab, bc, cd, &c. are all equal. Hence the time of reaching the highest point is infinite.

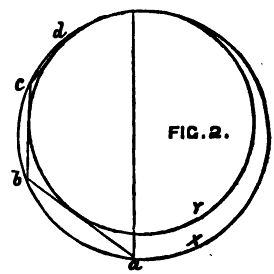


FIG. 2.

1598. (Proposed by Dr. BOOTH, F.R.S.)—If R and r be the coincident radii vectores of two inverse curves, so that $\mathrm{R}r = \text{a } constant = k^2$, and if C and c be the chords of curvature through the origin; then $\dfrac{\mathrm{R}}{\mathrm{C}} + \dfrac{r}{c} = 1$.

I. *Solution by* ARCHER STANLEY.

Since the circle of curvature passes through three consecutive points of the curve, and three points suffice to determine a circle, it follows that the circle of curvature at any point on the inverse curve is the inverse of that at

the corresponding point of the primitive. But if R, R′, r, r' be the distances from the origin to the intersections of any radius vector by two inverse circles, we have, from the definition of inverse curves,

$$\frac{R}{R'} = \frac{r'}{r}, \text{ whence } \frac{R}{R-R'} = -\frac{r'}{r-r'};$$

consequently, adding $\frac{r}{r-r'}$ to each side,

$$\frac{R}{R-R'} + \frac{r}{r-r} = 1,$$

which proves the theorem, if the inverse circles be supposed to be the circles of curvature at the corresponding points R, r.

II. *Solution by the* PROPOSER; X. U. J.; *and others.*

Let D be the diameter of curvature of one of the curves, p the perpendicular from the origin on the tangent; then $D = \frac{2r\,dr}{dp}$; and the cosine of the angle between the diameter of curvature and its chord passing through the origin is $\frac{p}{r}$; hence we have

$$c = 2p\frac{dr}{dp}, \quad \therefore \; \frac{r}{c} = \frac{r\,dp}{2p\,dr}; \quad \text{and similarly } \frac{R}{C} = \frac{R\,dP}{2P\,dR};$$

therefore $$2\left(\frac{r}{c} + \frac{R}{C}\right) = \frac{r\,dp}{p\,dr} + \frac{R\,dP}{P\,dR} \dots\dots\dots\dots\dots\dots (1).$$

Now, as $Rr = k^2$, $R\,dr + r\,dR = 0$; and as the tangents to the curves, at the points where they are cut by the common radius vector, are equally inclined to it, we have

$$\frac{p}{r} = \frac{P}{R}, \text{ or } P = \frac{Rp}{r} = \frac{k^2p}{r^2}; \quad \text{whence } \frac{dP}{P} = \frac{dp}{p} - \frac{2dr}{r};$$

but $$\frac{R}{dR} = -\frac{r}{dr}; \quad \therefore \; \frac{R\,dP}{P\,dR} = -\frac{r\,dp}{p\,dr} + 2 \dots\dots\dots\dots (2).$$

From (1) and (2) we have $\frac{R}{C} + \frac{r}{c} = 1.$

III. *Solution by* H. J. PURKISS, B.A.; *and* R. TUCKER, M.A.

The chord of curvature through the origin is equal to twice the radius of curvature multiplied by the sine of the angle between the tangent and radius vector;

hence $$c = \frac{2\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} \, \frac{r}{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}};$$

$$\text{therefore}\quad \frac{r}{c} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{2\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} \text{..................(1).}$$

$$\text{Now}\quad \frac{1}{R}(\equiv U) = \frac{r}{k^2};\quad \therefore\ C = \frac{2\left\{U^2 + \left(\frac{dU}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{U^3\left(U + \frac{d^2U}{d\theta^2}\right)}\ \frac{U}{\left\{U^2 + \left(\frac{dU}{d\theta}\right)^2\right\}^{\frac{1}{2}}};$$

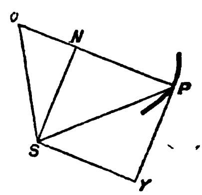
$$\therefore\ \frac{R}{C} = \frac{1}{CU} = \frac{U\left(U + \frac{d^2U}{d\theta^2}\right)}{2\left\{U^2 + \left(\frac{dU}{d\theta}\right)^2\right\}} = \frac{r\left(r + \frac{d^2r}{d\theta^2}\right)}{2\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} \text{............(2).}$$

Adding the results (1) and (2), we get $\frac{R}{C} + \frac{r}{c} = 1$.

1599. (Proposed by H. J. PURKISS, B.A.)—If S be the pole, P a point on a curve, O the centre of curvature at P, SP $= r$, and p = perpendicular from S on the tangent at P, prove that

$$\sin^2 PSO = \frac{r^2 - p^2}{r^2\left\{1 + \left(\frac{dp}{dr}\right)^2\right\} - 2pr\frac{dp}{dr}}.$$

Solution by the PROPOSER; X. U. J.; R. TUCKER, M.A.; *and others.*



Let SY be the perpendicular p, and SN perpendicular to OP; and let OP $= \rho$;

then $\qquad \sin PSO = \frac{OP}{OS} \sin SPO.$

Now $\qquad OP = \rho = r\frac{dr}{dp};$

$$OS^2 = SP^2 + PO^2 - 2PO \cdot PN = r^2 + \rho^2 - 2p\rho$$
$$= r^2\left\{1 + \left(\frac{dr}{dp}\right)^2\right\} - 2pr\frac{dr}{dp};$$

and $\qquad \sin SPO = \frac{\sqrt{(r^2 - p^2)}}{r}.$

Substituting and squaring, we get the required result.

1606. (Proposed by the EDITOR.) Through three given points to draw a conic whose foci shall lie in two given lines.

Solution by T. A. HIRST, F.R.S.

Professor CAYLEY, when treating this problem in the *Educational Times* for January, (*Reprint*, vol. ii. p. 99), was enabled to assign a limit to the number of its solutions. My present object is to find the precise number of solutions of the following equivalent problem :—

Through three given points to draw a conic which shall have, in common with a fixed conic C, two tangents intersecting on a given line A, and the remaining two on another line B.

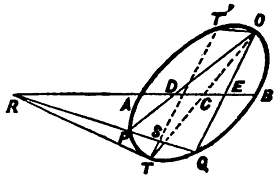
Of the conics S which pass through the three given points, there are, by a well known theorem, four which touch any two lines T and T′. (Salmon's *Conics*, p. 361, 4th ed., also Chasles' *Traité des Sections Coniques, p.* 42.) Hence we conclude that there are four conics S which touch an arbitrary line T, and have, with C, a common tangent T′. The latter will, of course, be cut in three points by the remaining tangents common to C and each such conic S; whence we may infer that the locus of the intersections of tangents common to C and a conic S, which touches any line T, is of the *twelfth* order. The intersections of this locus with a *perfectly arbitrary* line A will correspond, obviously, to twelve *distinct* conics S, which touch a line T, and have, in common with C, a pair of tangents intersecting on A. If T be itself a tangent to C, cutting A in *a*, then of the above twelve conics there will, as already remarked, be four which likewise touch the second tangent from *a* to C; consequently there will remain *eight* conics S which have, in common with C, the tangent T and two *other* tangents intersecting on A. The fourth common tangents to C and these conics S will intersect T in eight points; whence we infer that :—

If a conic S, passing through three fixed points, have in common with a fixed conic C two tangents which intersect on a given line A, the locus of the intersection of the remaining pair of common tangents will be a curve of the eighth order. Each of the intersections of this locus and the second given line B, corresponds to a solution of our problem, whence we conclude that the latter admits, in general, of *eight* solutions.

M. CHASLES' method of characteristics leads to the same result more quickly; the above solution has, in fact, at my request, been thus verified by Professor CREMONA.

1624. (Proposed by F. D. THOMSON, M.A.)—AB is a diameter of a conic; C its centre; P, Q any two points on the curve. It is required to find a point O on the curve, such that if OP, OQ meet AB in D and E, CD shall be equal to CE.

I. *Solution by the* PROPOSER.



Produce QP to meet AB in R; draw the tangent RT and the diameter TCO; then O will be the point required. Draw OT′ parallel to AB to meet the curve, and join TT′ meeting PQ in S. Then TT′ is the polar of R, since supplemental chords are parallel to conjugate diameters; therefore RPSQ is cut harmonically; hence we have

$$-1=\{QPSR\}=\{T\,.\,QPT'T\}=\{O\,.\,QPT'T\}=\{ED\propto C\}$$
$$=\left(\frac{E\infty}{\infty D}:\frac{EC}{CD}\right)=-\frac{CD}{EC};\ \therefore\ CD=CE.$$

II. *Solution by* ARCHER STANLEY.

The following more general question will be solved with equal readiness.

Any two points P and Q being given on a conic (C), to find a point O on the curve such that OP, OQ may intersect the connector of any two points a, b of the plane in D and E, so that the anharmonic ratio (abDE) shall have a given value λ.

If to each point p on ab the (unique) point q be determined by the given relation $(abpq)=\lambda$, it is well known that p and q will describe homographic ranges. The connectors Pp, Qq, therefore, will describe homographic pencils around P and Q, and their corresponding elements will intersect on a conic (Γ), which will manifestly intersect the given one in P, Q, and the required points O_1, O_2. There are two solutions, therefore, and the question is reduced to the well known problem of finding one of the common chords O_1O_2 to two conics (C) and (Γ) when the other PQ is known.

The curve (Γ), it will be observed, passes through a and b; moreover, in virtue of the relation $(abpq)=\lambda$, the two points p_1, q_2 can readily be found, whose corresponding points q_1, p_2 coincide with the intersection R of PQ, ab. This done, Pp_1 and Qq_2 will be the tangents to (Γ) in P and Q, so that their intersection r will be the pole of PQ relative to (Γ). Now the polars of r relative to all conics through the four intersections of (C) and (Γ) are concurrent (Salmon's *Conics*, p. 257, 4th ed.); and since the pair of lines PQ, O_1O_2 constitutes one of these conics, it is manifest that the polar of r relative to (C) will pass through the intersection R_2 of PQ and O_1O_2. The required chord O_1O_2, therefore, will be completely determined by another of its points; *e.g.* by its intersection R_1 with ab. To find R_1 it will be sufficient to remember that AB, ab, RR_1, where A and B are intersections (real or imaginary) of (C) and ab, constitute three pairs of points in involution (*ibid*. p. 297). A well known construction for the point R_1 in this involution, which is conjugate to the given point R, is as follows: Connect a and b with any point, say P, of the given conic (C) by lines which intersect the latter in α and β, respectively; and let $\alpha\beta$ cut AB in M. Then MQ will meet (C) again in ρ, so that Pρ will cut AB in the point R_1 required. The line R_1R_2 thus determined is always real, and whenever it actually intersects (C), it does so in the required points O_1 and O_2.

When ab is to be divided harmonically, that is to say, when $\lambda=-1$, the points p_1, q_2, r obviously coincide with the harmonic conjugate of R relative to ab, and the construction is greatly simplified. It is still simpler when A, B are likewise harmonic conjugates relative to ab; for then it may readily be shown that R_1 is precisely the harmonic conjugate of r relative to AB, so that R_1R_2 coincides with the polar of r relative to (C); in other words, *the required points* O_1, O_2 *are the intersections, with* (C), *of the polar, relative to this conic, of the harmonic conjugate* r, *with respect to the given points* a, b, *of the intersection* R *of* PQ *and* AB. In the Question, AB is a diameter, a coincides with the centre C of the conic, and b is at infinity; hence, if a point r be taken on AB so that $Cr=RC$, the polar of r, relative to the given conic, will intersect the latter in the required points O_1, O_2.

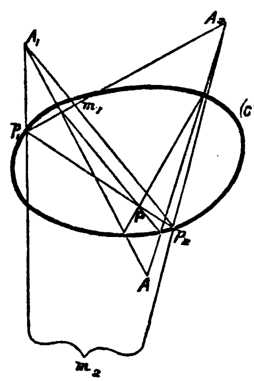
Two other special cases deserve notice. If a and b coincide with the

circular points at infinity, the problem resolves itself into the very simple one of finding the points on (C) at which PQ subtends a given angle, measured in a definite manner. The value $\lambda = -1$ now corresponds to the case of a right angle.

When a and b are at infinity in directions perpendicular to each other, and $\lambda = -1$, the problem assumes this form: *To find a point O on the given conic* (C) *such that the bisectors of the angle POQ may be parallel to two given perpendicular lines.* The construction in this case is simple, and especially so when the given perpendicular lines are the axes of (C); for then A, B being at infinity on (C) are again harmonic conjugates relative to ab. If c be the middle point of the segment intercepted by the axes on PQ, the extremities O_1O_2 of the diameter conjugate to Cc will be the points required; the point r being obviously at infinity on cC.

1646. (Proposed by the EDITOR.)—Through a given point to draw a chord of a conic, such that the straight lines joining its ends with two other given points shall contain a given angle.

Solution by ARCHER STANLEY.



If C^2 be the given conic, P the fixed point through which the variable chord p_1p_2 passes, and A_1, A_2 the two other fixed points, then the locus of the opposite vertices m_1, m_2 of the quadrilateral, formed by joining A_1, A_2, with p_1, p_2, will be a curve C^4 of the fourth order, which has double points at A_1, A_2 as well as at a third point A so situated, on the polar of P, that the chords through P, whose extremities are on one of the lines AA_1, AA_2, have their opposite extremities on the other. The existence of this point A is most readily seen by projecting C^2 and P into a circle and its centre. That A_1 and A_2 are double points on the locus C^4 will be evident on joining A_1A_2, and drawing the chords through P, to its intersections (real or imaginary) with C^2. It will then be seen that the two points m_1m_2, corresponding to each of these chords, coincide with A_1, A_2. This granted, the locus C^4 is seen to be of the fourth order; in fact, every line through A_1 must meet it in two points, exclusive of A_1 itself, for such a line intesects C^2 in two points, and these determine two chords through P whose opposite extremities joined to A_2, give two other lines intersecting the first in points on the locus C^4. From this, it is manifest that A is a double point, since A_1A, as well as A_2A, cut the locus C^4 in two points coincident with A. Now the locus of a point m such that the angle A_1mA_2, estimated in a determinate manner, shall be equal to a given one, is

a circle M^2 intersecting the quartic C^4 in *four* points (exclusive of the double points A_1, A_2), each of which, when joined to A_1A_2, will determine a chord through P of the required kind. Should the given angle be equal to A_1AA_2 the point A itself will lead to two solutions, and only two others will remain. The number of solutions will of course be doubled if, in estimating the given angle, the direction of rotation required in order to bring A_1m into parallelism with A_2m is not stated.

If it were required to draw a chord p_1p_2 so that the corresponding angle A_1mA_2 should be a maximum or minimum, it would be necessary to draw a circle M^2 through A_1, A_2 so as to *touch* the quartic C^4 elsewhere. By a known method of transformation (Salmon's *Higher Plane Curves*, Art. 259), the number of such circles may be shown to be equal to the number of conics, passing through four points, which touch a given conic, and this is well known to be *six*. (Salmon's *Conics*, 4th Ed. Art. 388, Ex. 1.)

When A_1, A_2 are on the given conic C^2 the question may be more simply investigated thus:—

Any ray through A_1 will meet C^2 again in a point a_1, and a_1P will intersect C^2 a second time in a point a_2 which, connected with A_2, gives a ray A_2a_2 corresponding to A_1a_1. Now to every ray A_1a_1 corresponds but one ray A_2a_2, and *vice versâ;* hence, by a well known principle, the rays A_1a_1, A_2a_2 correspond anharmonically, and generate by their intersection m a conic Σ^2 passing through A_1A_2. Further A_1P, A_2P are the tangents to Σ^2 at A_1, A_2; since they are the rays which, in their respective pencils, obviously correspond to A_1A_2. If A_1P, A_2P cut the conic again in a_1, a_2, respectively, A_1a_2, A_2a_1 will correspond, respectively, to the tangents, to C, at A_2, A_1; we have thus two more points on the conic Σ^2, and it may be observed that the latter, and the right lines A_1a_2, A_2a_1 (meeting in A on the polar of P) constitute, in the present case, the quartic C^4. The circle M^2 through A_1A_2 meets the conic Σ^2 in two other points m, such that A_1m, A_2m not only enclose the given angle, but intersect C^2 again in two points a_1, a_2 collinear with P. If A_1, A_2 and P were collinear, Σ^2 would break up into A_1A_2 and another line L. We should still, however, have two solutions of our Question.

1595. (Proposed by M. W. Crofton, B.A.)—A uniform beam rests in a given oblique position between two parallel vertical walls, just supported by the friction: the two coefficients of friction are given, and both ends are on the point of slipping independently. Determine the directions in which the frictions act at each end, and show that a certain relation must hold between the two coefficients of friction.

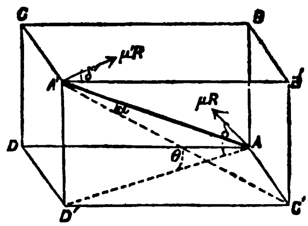
Solution by the Proposer.

Let α be the inclination of the beam to the walls, θ the inclination of the vertical plane through the beam to the walls; then δ, δ', the angles which the frictions make with the horizon, are given by

$$\mu \cos \delta = \mu' \cos \delta' = \cot \theta,$$

and the relation between μ and μ' is

$$(\mu^2-\mu'^2)^2 = 8(\cot^2\alpha-\cot^2\theta)(\mu^2+\mu'^2-2\cot^2\alpha).$$



Let AA′ be the beam, ABCD, A′B′C′D′ the walls: complete the parallelopiped AA′, of which the walls form two opposite faces. The normal pressure (R) is the same at A and A′; resolve the frictions μR, μ'R horizontally and vertically. Now, by resolving the weight (W) of the beam into two ($\frac{1}{2}$W) acting at A, A′, we may suppose the beam without weight, and we shall have three rectangular forces acting at A, viz., R in AC′, $\mu R\cos\delta$ in AD, $\mu R\sin\delta-\frac{1}{2}W$ in AB. These must give a resultant along the beam, and therefore are as the edges of the parallelopiped in which they act. The same is true at the point A′, the three forces there being R, $\mu' R\cos\delta'$, $\frac{1}{2}W-\mu' R\sin\delta'$.

Hence $$\frac{\mu R\cos\delta}{R} = \frac{AD}{AC'} = \cot\theta;$$

so that the directions of the frictions are given by

$$\mu\cos\delta = \mu'\cos\delta' = \cot\theta \quad\ldots\ldots\ldots\ldots(1).$$

Also $$\frac{(\mu R\sin\delta-\frac{1}{2}W)^2}{R^2} = \frac{AB^2}{AC'^2} = \frac{A'C'^2-C'D'^2}{AC'^2} = \cot^2\alpha-\cot^2\theta;$$

$$\therefore\quad \mu\sin\delta-\frac{W}{2R} = \sqrt{(\cot^2\alpha-\cot^2\theta)}. \quad \text{Also}\quad W = R(\mu\sin\delta+\mu'\sin\delta').$$

Hence, eliminating $\frac{W}{R}$, we have $\mu\sin\delta-\mu'\sin\delta' = 2\sqrt{(\cot^2\alpha-\cot^2\theta)}$,

or, by (1), $$\sqrt{(\mu^2-\cot^2\theta)}-\sqrt{(\mu'^2-\cot^2\theta)} = 2\sqrt{(\cot^2\alpha-\cot^2\theta)},$$

or $$(\mu^2-\mu'^2)^2 = 8(\cot^2\alpha-\cot^2\theta)(\mu^2+\mu'^2-2\cot^2\alpha)\ldots\ldots\ldots\ldots(2),$$

an equation which μ, μ' must satisfy.

1593. (Proposed by H. R. Greer, B.A.)—(1.) Given a conic and two lines, find the (trilinear) coordinates of the point where the polar of the intersection of these meets an assigned one of them.

(2.) Hence determine, in trilinear coordinates, the form of the equation of a line perpendicular to a given line.

(3.) Determine *directly* the locus of a point such that if perpendiculars be drawn from it to the sides of a triangle, their feet shall lie in a straight line.

(4.) What does this proposition become by a general homographic transformation?

Solution by the Proposer.

Let $u=0$ be the conic; $ax+by+cz=0$, $lx+my+nz=0$ the given lines; and let it be required to find the point where the *former* of these is

met by the polar (with respect to the conic) of the intersection of the pair. This may be done by eliminating x, y, z from the equations

$$ax + by + cz = 0,\ lx + my + nz = 0,\ \frac{du}{dx'} \cdot x + \frac{du}{dy'} \cdot y + \frac{du}{dz'} \cdot z = 0,$$

x', y', z' being the current coordinates of the polar, and then finding $x' : y' : z'$ from the equations $ax' + by' + cz' = 0$ and the above eliminant. Now let the conic be the circumscribing circle, viz., $ayz + bzx + cxy = 0$; a, b, c being the sides of the triangle of reference; then the concluding equations will be

$$\begin{vmatrix} bz' + cy', & cx' + az', & ay' + bx' \\ l\ , & m\ , & n \\ a\ , & b\ , & c \end{vmatrix} = 0, \text{ and } ax' + by' + cz' = 0;$$

from which we find x' proportional to

$$l\,(2abc) - mc\,(a^2 + b^2 - c^2) - nb\,(c^2 + a^2 - b^2);$$

and similar expressions for y' and z'; that is, dividing by $2abc$, we have

$$x' : y' : z' = l - m \cos C - n \cos B : m - n \cos A - l \cos C : n - l \cos B - m \cos A.$$

These expressions are therefore proportional to the coordinates of a point at infinity in a direction perpendicular to $lx + my + nz = 0$.

I may remark that the analogous propositions in space are as follows:

Given a conicoid and two planes, find the coordinates of the point where the reciprocal polar line (with respect to the conicoid) of the line of intersection of these meets an assigned one of them.

And by taking as the conicoid any sphere (say that circumscribing the tetrahedron of reference) and as the "assigned" plane of the two the plane at infinity, we can determine in like manner expressions proportional to the coordinates of a point at infinity in a direction perpendicular to a given plane.

Now let $lx + my + nz = 0$ represent, successively, the three sides of the triangle of reference; then from the formulæ just given, we find the coordinates of the feet of the perpendiculars drawn from an arbitrary point (f, g, h) on the sides of the triangle of reference, to be as follows: viz.,

$$x = 0,\ y = g + f \cos C,\ z = h + f \cos B, \text{ on the side } x,$$
$$y = 0,\ z = h + g \cos A,\ x = f + g \cos C, \text{ on the side } y,$$
$$z = 0,\ x = f + h \cos B,\ y = g + h \cos A, \text{ on the side } z.$$

The condition that these three points should lie in a straight line is

$$\begin{vmatrix} 0\ , & g + f \cos C, & h + f \cos B \\ f + g \cos C, & 0\ , & h + g \cos A \\ f + h \cos B, & g + h \cos A, & 0 \end{vmatrix} = 0.$$

This determinant is, when worked out,

$$(f + g \cos C)\,(g + h \cos A)\,(h + f \cos B) + (h + g \cos A)\,(f + h \cos B)\,(g + f \cos C),$$

which breaks up identically into the product of

$$f \sin A + g \sin B + h \sin C \quad \text{into} \quad gh \sin A + hf \sin B + fg \sin C;$$

this result determining the locus of (f, g, h), according to what we all know, to be the circle circumscribing the original triangle.

The geographical statement of this theorem homographically transformed, runs thus: Consider two fixed points joined by a line intersecting the three sides of a triangle; to each point of intersection successively let there be

taken the harmonic conjugate with the fixed points as a pair of conjugates: join each point so determined with any point on the conic passing through the fixed points and the vertices of the triangle; the intersections of these connectors each with the "corresponding" side of the original triangle, lie in a straight line. And, analytically stated, the proposition is as follows: Let (α, β, γ), $(\alpha', \beta', \gamma')$ be the two fixed points, and (f, g, h) the arbitrary point on the conic circumscribing the triangle of reference. The coordinates of the "harmonic conjugates" spoken of above are as follows; viz.,

$$x : y : z = \gamma'\alpha + \gamma\alpha' : \gamma'\beta + \gamma\beta' : 2\gamma\gamma'$$

for that corresponding to the point of intersection on the side z, and similar expressions for the other two. The coordinates of the point where the line joining *this* conjugate with the point (f, g, h) intersects the side z, are

$$x : y : z = h(\gamma'\alpha + \gamma\alpha') - 2f\gamma\gamma' : h(\gamma'\beta + \gamma\beta') - 2g\gamma\gamma' : 0,$$

with similar expressions for the other two points of intersection; and the condition that these three should lie in a straight line is

$$\begin{vmatrix} 0, & f(\alpha'\beta + \alpha\beta') - 2g\alpha\alpha', & f(\alpha'\gamma + \alpha\gamma') - 2h\alpha\alpha' \\ g(\beta\alpha' + \beta'\alpha) - 2f\beta\beta', & 0, & g(\beta'\gamma + \beta\gamma') - 2h\beta\beta' \\ h(\gamma'\alpha + \gamma\alpha') - 2f\gamma\gamma', & h(\gamma'\beta + \gamma\beta') - 2g\gamma\gamma', & 0 \end{vmatrix} = 0.$$

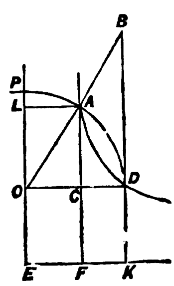
And this condition ought to be fulfilled only when f, g, h, lies upon the circumscribing conic passing through the two fixed points, or upon the line joining the two fixed points; hence, finally, the analytical theorem we arrive at is, that the foregoing determinant should break up into the product of the two determinants

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ f, & g, & h \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \beta\gamma, & \gamma\alpha, & \alpha\beta \\ \beta'\gamma', & \gamma'\alpha', & \alpha'\beta' \\ gh, & hf, & fg \end{vmatrix},$$

a result which I have not sought to verify.

1612. (Proposed by J. O'CALLAGHAN.) — Through the centre of a given circle to draw a secant, such that the part of it intercepted between the circumference and a fixed tangent may have a given ratio to the sine of the intercepted arc.

Solution by ALPHA; *the* PROPOSER; E. FITZGERALD; *and others.*



Let O be the centre of the given circle, and BD the fixed tangent. Produce BD to K, making OD : DK equal to the given ratio; complete the rectangle ODKE; and through D, along the asymptotes EK, EO, draw an equilateral hyperbola cutting the circle in A; then OAB will be the secant required.

For complete the rectangle ELAF; then, by the property of the hyperbola, ODKE = ELAF, and CDKF = CALO;

therefore AC : CF = DC : CO = BA : AO;

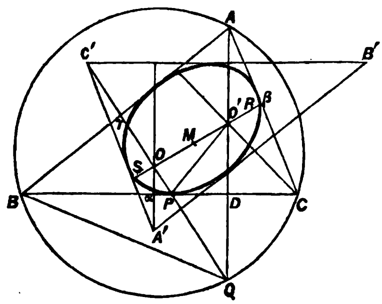
also OD : AC = AO : AC;

hence, compounding these ratios, we have

BA : AC = OD : DK = *the given ratio.*

1618. (Proposed by J. GRIFFITHS, M.A.)—Let α, β, γ be the middle points of the sides of any triangle ABC; O′ the point of intersection of its three perpendiculars, and O the centre of its circumscribing circle. Produce $O\alpha$, $O\beta$, $O\gamma$ to A′, B′, C′, so that $OA' = 2O\alpha$, $OB' = 2O\beta$, $OC' = 2O\gamma$. It is required to prove:—(1) That the sides of the triangles ABC, A′B′C′ are touched by the same conic. (2) That the points O, O′ are the foci of this conic, and that its major axis is equal to the radius of circle circumscribing either of these triangles. (3) That the common nine-point circle of the two triangles is the auxiliary circle of the conic.

Solution by the EDITOR.



Several properties of two such triangles have been proved in the Solution of Question 1383 (see p. 6, Vol. I. of the *Reprint.*)

It is there shown that each of the triangles may be obtained in precisely the same way from the other; that the intersections (O, O′) of the perpendiculars of either of the two triangles is the centre of the circle drawn round the other, and *vice versâ*; and that the two triangles have their sides parallel, are in all respects equal to each other, and have a common nine-point circle, the centre of which is, of course, at the middle point M of OO′, and its radius equal to half that (OA, O′A′, &c.) of the circle drawn round either of the triangles ABC, A′B′C′.

The properties in the Question may hence be readily proved. For let the perpendicular AD meet the circle ABC in Q; draw OQ, cutting BC in P; and join O′P. Then $\angle QBD = DAC = DBO'$, whence it follows that $O'D = DQ$, and $O'P = PQ$; therefore $OP + PO' = OQ$, and $\angle OPB = QPD = O'PC$.

If, therefore, a conic (an *ellipse* in the figure) be drawn with O, O′ as foci, and RS ($= OP + PO' = OQ = OA$, &c.) as major axis, it will *touch* BC at P; and the like may be proved with respect to the other sides of the triangles ABC, A′B′C′. Since, moreover, the common nine-point circle of the triangles is concentric with the ellipse (M being the common centre), and its diameter is equal to the major axis of the ellipse, it must be the auxiliary (or circumscribing) circle of the ellipse.

When ABC is an *acute-angled* triangle (as in the above diagram), the tangential conic is an *ellipse*; when ABC is *right-angled*, the conic becomes two coincident straight lines (an indefinitely flattened ellipse); and when ABC is *obtuse-angled* (as in the figure to Quest. 1383), the points O, O′ are outside both triangles, and the conic is a *hyperbola*.

1577. (Proposed by the Rev. J. BLISSARD.)—Prove the following properties of numbers :—

$$(1)\ldots.\ \frac{1}{1}-\frac{1}{2}+\frac{1}{3}\ldots\ldots-\frac{1}{n}\ (n \text{ even}) = 2\left(\frac{1}{n+2}+\frac{1}{n+4}+\ldots.+\frac{1}{2n}\right),$$

$$(2)\ldots.\ \frac{1}{1}-\frac{1}{2}+\frac{1}{3}\ldots\ldots+\frac{1}{n}\ (n \text{ odd}) = 2\left(\frac{1}{n+1}+\frac{1}{n+3}+\ldots.+\frac{1}{2n}\right).$$

Solution by S. BILLS; *the* PROPOSER; *and others.*

1. Suppose the theorem to be true for any specified even number (n) of terms, that is, suppose

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}\ldots.-\frac{1}{n}=2\left(\frac{1}{n+2}+\frac{1}{n+4}+\ldots.+\frac{1}{2n}\right);$$

then, taking in two more terms, we shall have

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}\ldots.-\frac{1}{n}+\frac{1}{n+1}-\frac{1}{n+2}=2\left(\frac{1}{n+2}+\ldots.\frac{1}{2n}\right)+\frac{1}{n+1}-\frac{1}{n+2}$$

$$=2\left(\frac{1}{n+4}+\ldots.\frac{1}{2n}\right)+\frac{1}{n+1}+\frac{1}{n+2}$$

$$=2\left(\frac{1}{n+4}+\frac{1}{n+6}+\ldots\frac{1}{2n+2}+\frac{1}{2n+4}\right)$$

Hence if the theorem holds for one even value of n, it holds for the next, and so on. But it does hold when $n=2$; therefore it holds for $n=4$, 6, &c., that is, *generally* for any even value of n.

2. In precisely the same manner it may be shown that if

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}\ldots.+\frac{1}{n}\,(n \text{ odd})=2\left(\frac{1}{n+1}+\frac{1}{n+3}+\ldots.\frac{1}{2n}\right)$$

holds for one particular odd value of n, it holds for the next odd value; but it does hold for $n=1$; therefore it holds for $n=3$, 5, &c., that is, generally, for any odd value of n.

1600. (Proposed by the EDITOR.)—Eliminate h and k from the equations

$$x^2+(y-k)^2+2x\,(y-k)\cos\alpha = a^2 \ldots\ldots..(1),$$
$$y^2+(x-h)^2+2y\,(x-h)\cos\alpha = b^2 \ldots\ldots..(2),$$
$$h^2+k^2-2hk\cos\alpha = c^2 \ldots\ldots..(3);$$

and express the result as a *rational* function of x and y.

Solution by S. BILLS.

Let r denote the cosine and s the sine of α. Then from (1) and (2) we readily find

$$k=y+rx+(a^2-s^2x^2)^{\frac{1}{2}},\ h=x+ry+(b^2-s^2y^2)^{\frac{1}{2}}.$$

Substituting these results in (3), reducing, and putting $m = \frac{1}{2}(a^2+b^2-c^2)$, we have

$$s^2\left\{y(a^2-s^2x^2)^{\frac{1}{2}}+x(b^2-s^2y^2)^{\frac{1}{2}}\right\}=r(a^2-s^2x^2)^{\frac{1}{2}}(b^2-s^2y^2)^{\frac{1}{2}}-(m+rs^2xy)\,..(4).$$

Squaring (4), and putting $n^2=m^2+r^2a^2b^2$, we obtain

$$2(rm+s^2xy)(a^2-s^2x^2)^{\frac{1}{2}}(b^2-s^2y^2)^{\frac{1}{2}}=n^2+2s^4x^2y^2-s^2(a^2y^2+b^2x^2)$$
$$+2rs^2mxy\,\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Now put $s^2(a^2y^2+b^2x^2)=u$ and $s^2xy=v$, then (5) becomes, after squaring, &c.,

$$4(rm+v)^2(a^2b^2-u+v^2)=(n^2+2v^2-u+2rmv)^2,$$

or, $\quad u^2+2(2rmv+2r^2m^2-n^2)u=$

$$4(a^2b^2-n^2)v^2+4rm(2a^2b^2-n^2)v-(m^2-r^2a^2b^2)^2\,\ldots\ldots\ldots\ldots(6).$$

Solving (6) we obtain

$$u=n^2-2rm(v+rm)\pm 2s(a^2b^2-m^2)^{\frac{1}{2}}(v+rm),$$

or, restoring the values of u and v, we have, finally,

$$s^2(a^2y^2+b^2x^2)=n^2-2r^2m^2-2rs^2mxy\pm 2s(a^2b^2-m^2)^{\frac{1}{2}}(s^2xy+rm)\ldots\ldots\ldots(7).$$

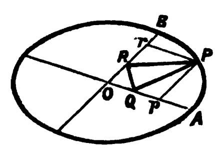
[NOTE.—If, in the foregoing equations, a, b, c denote the sides of a triangle ABC, we shall have $m=ab\cos C$, $(a^2b^2-m^2)^{\frac{1}{2}}=ab\sin C$, and the resulting equation (7) will then become

$$b^2x^2+2ab\cos(\alpha\pm C)\,.\,xy+a^2y^2=a^2b^2\sin^2(\alpha\pm C)\operatorname{cosec}^2\alpha\ldots\ldots\ldots(8).$$

Thus, if a triangle ABC move with its vertices A and B on two fixed straight lines (OX, OY) including an angle α, and if, in reference to these lines as axes, (x, y) be the coordinates of C, also OA $= h$ and OB $= k$; then the above elimination contains an investigation of the locus of the vertex C; and the resulting equation (8) shows that this locus is an *ellipse* around the centre O, which degenerates into a straight line when $\cos(\alpha\pm C)=\pm 1$. EDITOR.]

1580. (Proposed by S. BILLS.)—To place a given triangle in a given ellipse so that the vertices shall be situated, one in each of two given conjugate diameters, and the third in the curve.

Solution by the PROPOSER.



In the accompanying diagram, let O be the centre of the given ellipse APB; OA, OB, the two given conjugate semi-diameters; and PQR the position of the given triangle. Draw Pp, Pr, parallel, respectively, to OB, OA. If OA $= f$, OB $= g$, the equation of the ellipse will be

$$\frac{x^2}{f^2}+\frac{y^2}{g^2}=1\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

Put RP $= a$, PQ $= b$, QR $= c$, OQ $= h$, OR $= k$, O$p = x$, O$r = y$, and $\angle$ AOB $= \alpha$. Then we have the following system of equations; viz.,

$$x^2 + (y-k)^2 + 2x\,(y-k)\cos\alpha = a^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$y^2 + (x-h)^2 + 2y\,(x-h)\cos\alpha = b^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$h^2 + k^2 - 2hk\cos\alpha = c^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Eliminating h and k from (1), (2), (3), the result is (see Question 1600),

$$s^2\,(a^2y^2 + b^2x^2) = n^2 - 2r^2m^2 - 2rs^2mxy \pm 2s\,(a^2b^2 - m^2)^{\frac{1}{2}}\,(s^2xy + rm),$$

or, say, $$a^2y^2 + b^2x^2 + m_1 = n_1xy \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B),$$

where $$m_1 = \frac{1}{s^2}\left\{\pm 2rsm\,(a^2b^2 - m^2)^{\frac{1}{2}} - n^2\right\},\; n_1 = \pm 2s\,(a^2b^2 - m^2)^{\frac{1}{2}} - 2rm.$$

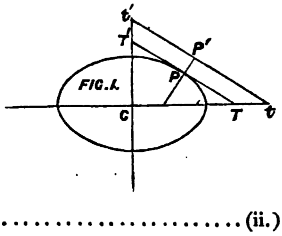
Squaring (B), and substituting therein the value of x^2 deduced from (A), we readily obtain y by a quadratic equation; and thence the *positions* of the given triangle PQR will be completely determined.

[Note.—By referring to the *Note* at the end of the foregoing Question 1600, it will be seen that the positions of P may be determined by the intersections of the given ellipse with a concentric ellipse which is the locus of the vertex P of the triangle PQR, supposing the triangle to move with the vertices Q and R on the lines OA, OB.—Editor.]

1558. (Proposed by Dr. Booth, F.R.S.)—The normals to an ellipse are elongated by a constant quantity k, measured from the curve: show that the tangential equation of the curve thus generated, which may be called the *parallel* to the ellipse, is $\left\{(a^2-k^2)\,\xi^2 + (b^2-k^2)\,v^2 - 1\right\}^2 = 4k^2\,(\xi^2 + v^2)$, and prove that the length of the parallel curve is equal to that of the elliptic base together with that of a circle whose radius is k.

Solution by F. D. Thomson, M.A.

1. Let TPT′ be a tangent to the ellipse, tP′t' the corresponding tangent to the parallel curve, so that PP′ $= k$.



Let $\angle$ PTC $= \theta$; then, if

$$\text{CT} = \frac{1}{\xi'},\; \text{CT}' = \frac{1}{v'},\; \text{C}t = \frac{1}{\xi},\; \text{C}t' = \frac{1}{v};$$

we have for the equation of the ellipse

$$a^2\xi'^2 + b^2v'^2 = 1 \;\ldots\ldots\ldots\ldots\ldots (\text{i.})$$

Also $\text{C}t = \text{CT} + \dfrac{k}{\sin\theta}$, or $\dfrac{1}{\xi} = \dfrac{1}{\xi'} + \dfrac{k}{\sin\theta}$ $\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (ii.)

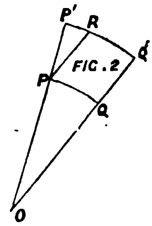
Now $\tan\theta = \dfrac{\text{C}t'}{\text{C}t} = \dfrac{\xi}{v}$; hence, from (ii), $\dfrac{1}{\xi} = \dfrac{1}{\xi'} + \dfrac{k\sqrt{(\xi^2+v^2)}}{\xi}$;

$$\therefore\; \xi'\left\{1 - k\sqrt{u^2+v^2)}\right\} = \xi. \quad \text{Similarly } v'\left\{1 - k\sqrt{(\xi^2+v^2)}\right\} = v.$$

Hence by (i.) we have $\left\{1-k\sqrt{(\xi^2+v^2)}\right\}^2 = a^2\xi^2+b^2v^2$,

$\therefore\ \left\{(a^2-k^2)\,\xi^2+(b^2-k^2)\,v^2-1\right\}^2 = 4k^2\,(\xi^2+v^2)$, the equation required.

2. For the second part of the Question, let PQ be an element of the ellipse, and P′Q′ the corresponding element of the parallel curve. Draw PR parallel to QQ′, then ultimately PR is perpendicular to P′Q′, and element $P'Q' = PQ + P'R = PQ + k\delta\phi$, if $\delta\phi$ is the angle between the normals at P and Q; therefore the whole length of the outer curve = length of ellipse + $2\pi k$.



[NOTE.—The Proposer's Solution is as follows:—Let P, p be the perpendiculars from C on tt', TT′ respectively, then $p^2 = a^2\sin^2\theta + b^2\cos^2\theta$ in the ellipse; and as the tangents are parallel, $\sin^2\theta = P^2\xi^2$, $\cos^2\theta = P^2v^2$, and $P^2 = (\xi^2+v^2)^{-1}$; hence substituting for P, p these values in the relation $p = P - k$, and reducing, we get the required equation of the parallel curve.—EDITOR.]

NOTE *by the* EDITOR *on Question* 1577. (*See p.* 54.)

These properties of numbers were given by Mr. J. R. YOUNG (formerly Professor of Mathematics at Belfast) in Quest. 1332 (*Educational Times* for Dec. 1862) under the following slightly different form of enunciation:—

"Prove that any *even* number of terms of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ is equal to the latter half of those terms, all taken positive."

The following proof is due to the Rev. R. HARLEY, F.R.S.:—

Write $F(2n) = 1-\frac{1}{2}+\frac{1}{3}\ldots\ldots-\frac{1}{2n}$, and $\phi(n) = 1+\frac{1}{2}+\frac{1}{3}\ldots\ldots+\frac{1}{n}$;

then $$F(2n)+\phi(2n) = 2\left(1+\frac{1}{3}+\frac{1}{5}+\ldots\ldots\frac{1}{2n-1}\right),$$

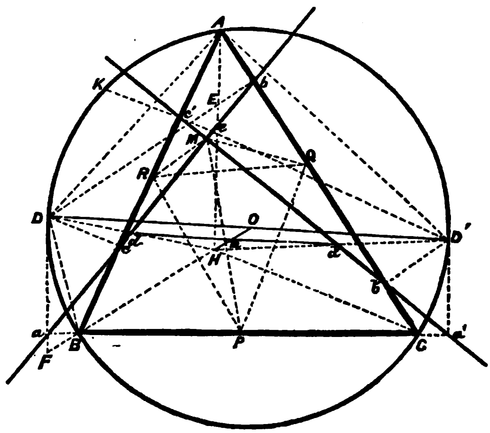
and $$2F(2n) = 2\left(1+\frac{1}{3}+\frac{1}{5}\ldots\ldots+\frac{1}{2n-1}\right)-\left(1+\frac{1}{2}+\frac{1}{3}\ldots\ldots+\frac{1}{n}\right)$$
$$= F(2n)+\phi(2n)-\phi(n);$$

whence $F(2n) = \phi(2n)-\phi(n)$, which establishes the theorem.

Mr. YOUNG remarks that the property may be used to obtain, in a readier way than otherwise, the sum of any number of terms of the proposed series; since, to obtain the sum of an even number $(2n)$ of terms of it we should only have to sum the last n of those terms, all positive, and for an odd number $(2n+1)$ the last $(n+1)$.

1649. (Proposed by R. TUCKER, M.A.)—From the ends of a diameter of a given circle perpendiculars are drawn on the sides of an inscribed triangle, prove that the two feet-perpendicular lines intersect at right-angles on the nine-point circle of the triangle.

I. *Solution by* ARCHER STANLEY.



Let H be the intersection of the perpendiculars of the triangle ABC, O the centre of its circumscribed circle, and D, D′ the opposite extremities of any diameter of the latter. Then Da, Db, Dc being perpendiculars from D on the sides of the triangle, a circle may obviously be drawn round each of the quadrangles DcAb, DcBa; the angles Acb, Bca in these circles are of course equal to ADb, BDa, respectively; and the latter are equal to one another, since, on adding BDb to each, we obtain the supplement of C. Hence may be deduced, not only the collinearity of a, b, c, but also the similarity of the triangles ADb, BDa; so that Db : bA = Da : aB. Moreover if HA, HB intersect Db, Da in E, F respectively, the angles at E and F of the parallelogram DEHF will be equal, and AEb, BFa will also be similar triangles; so that bA : Eb = aB : Fa. Hence, *ex equali*, Db : Eb = Da : Fa. But if HA intersect abc in e, Db : Eb = Da : Ee, whence we infer that Fa = Ee, and hence that abc passes through the centre of the parallelogram DEHF, that is to say bisects HD in d. Similarly D′a', D′b', D′c' being the perpendiculars from D′ on the sides of ABC, the line $a'b'c'$ of their feet bisects HD′ in d'. Furthermore dd', which is manifestly parallel and equal to either half of DD′, bisects and is bisected by HO in m, the centre of the nine-point circle (m). Consequently, the linear dimensions of the latter being half those of (O), *d and d' must be opposite extremities of a diameter of* (m).

Again, M being the intersection of the lines abc, $a'b'c'$, the angle Maa' is the complement of cad, which latter is equal to cBD, and this again to AD′D; so that the angles ADD′ and Maa' are equal to one another; as are also, for a similar reason, the angles AD′D and M$a'a$. But if so, then the angle aMa' will obviously be equal to the *right angle* DAD′, and M will lie with d and d' on the nine-point circle (m).

The above Solution, though not the most direct one of the proposed Question, prepares the way, in some measure, for investigating geometrically the envelope of the line abc. (Quest. 1668.) This envelope, I may observe, forms the subject of one of *Steiner's* papers in Crelle's *Journal.* (Vol. 53.) Numerous properties of the curve are given, but no demonstrations.

II. *Solution by the* Rev. R. TOWNSEND, M.A.; *and the* PROPOSER.

Since DbA, DcA, $D'b'A$, $D'c'A$ are all right angles, the angles Dbc, $Dc'b'$ an equal, respectively, to DAc, $D'Ab'$; and DAD' is a right angle; therefore bMc' is also a right angle.

Again, let Q, R be the middle points of AC, AB, and therefore also of bb', cc'; then the angles QMb', RMc are equal respectively to $Qb'M$, RcM, consequently the angle QMR is supplemental to the angle BAC, or QPR, P being the middle point of BC; hence M lies on the circle through P, Q, R, that is to say, on the nine-point circle of the triangle ABC.

III. *Solution by* H. R. GREER, B.A.; F. D. THOMSON, M.A.; E. FITZGERALD; J. DALE; *and others.*

Lemma 1. In a circle conceive any chord and any diameter; from one end of the latter let a perpendicular to the chord be drawn, and produced to meet the circle again; then the arc intercepted between this last so-determined point and either end of the chord is equal to the arc between the other end of the chord and the other end of the diameter, viz., that from which the perpendicular has *not* been drawn. Also, perpendiculars drawn from *both* ends of the diameter intercept equal portions on the chord, measured from the circumference.

Lemma 2. If two right-angled triangles have their sides coincident in direction, the angle between their hypotenuses is equal to that between the lines drawn from the common vertex of the triangles to the middle points of the hypotenuses.

Now, in the above figure, let K be the point in which $D'c'$ meets the circle again; then, by Lemma 1, the arc $AD = BK$, and the angle $DBc = BD'c'$; but $DBc = Dac$, and $BD'c' = Ba'c'$; which clearly shows abc to be perpendicular to $a'b'c'$, on observing that DaB is a right angle.

Again, consider the triangles aMa', cMc'; then $Ac' = Bc$, and $Ba = Ca'$, by Lemma 1; hence the middle points P, R of aa', cc', are likewise the middle points of BC, BA; consequently, by Lemma 2, the angle $PMR = PBR = PQR$, and M lies on the nine-point circle PQR of the triangle ABC.

1615. (From the *London University Examination Papers.*)—From a point taken at random inside a spherical surface of radius a, a straight line of length c is drawn at random. Find the chance that the straight line will intersect the surface. If $c = \frac{1}{2}a$, prove that the chance is $\frac{47}{128}$.

Solution by EDWARD FITZGERALD.

The number of indefinitely small right solids that can be taken in a sphere of radius a is proportional to $\frac{4}{3}\pi a^3$. The number of indefinitely small areas that can be taken on the surface of a sphere of radius c is proportional to

$4\pi c^2$. If, therefore, from a point in each of the indefinitely small solids a sphere, of radius c, be drawn, and its surface divided into indefinitely small areas, the total number of such areas on all the spheres thus drawn is proportional to $\frac{16}{3}\pi^2 a^3 c^2$. Therefore the total number of lines, of length c, that can be drawn from a point in each of the indefinitely small solids to the indefinitely small areas is proportional to $\frac{16}{3}\pi^2 a^3 c^2$.

Again, the number of indefinitely small solids in the spherical shell whose radius is $(a-c+x)$ and thickness dx is proportional to $4\pi\,(a-c+x)^2\,dx$. Also if a sphere of radius c be drawn from any point in one of these solids as centre, the part of its surface without the given sphere will be equal to $\pi c\,\dfrac{2ax+x^2}{a-c+x}$. Therefore the entire number of lines, of length c, that can be drawn from all points of the spherical shell to the surface of the given sphere is

$$4\pi^2 c\,(a-c+x)\,(2ax+x^2)\,dx\,;$$

and the *total* number of such lines that can be drawn to meet the surface of the given sphere is

$$4\pi^2 c\int_0^c (a-c+x)\,(2ax+x^2)\,dx = \tfrac{1}{3}\pi^2 c^3\,(12a^2-c^2).$$

Therefore the chance required $= \dfrac{\frac{1}{3}\pi^2 c^3\,(12a^2-c^2)}{\frac{16}{3}\pi^2 a^3 c^2} = \dfrac{c\,(12a^2-c^2)}{16a^3}$.

Putting $c = \lambda a$, the chance is $\frac{3}{4}\lambda - \frac{1}{16}\lambda^3$, which, when $\lambda = \frac{1}{2}$, becomes $\frac{47}{128}$.

1647. (Proposed by Professor CAYLEY.)—Find the locus of the foci of an ellipse of given major axis, passing through three given points.

[In connexion with the problem the Proposer remarks as follows:—

Let A, B, C be the given points; take P an arbitrary point (not in general in the plane of the three given points), then we may find a point Q (not in general in the plane of the three given points) such that $QA + AP = QB + BP = QC + CP =$ given major axis. And this being so, if the locus of P be a given surface, then we shall have a certain surface, the locus of Q; and so if the locus of P be a given curve in space, then we shall have a given curve in space, the locus of Q. In particular, if the locus of P be the plane of the three given points, then the locus of Q will be a certain surface, cutting the plane in a curve which is the locus in the foregoing problem; and when Q is situate on this curve, then also P will be situate on the same curve. Or if the locus of P be the curve in question, then the locus of Q will be the same curve. Say, in general, that the loci of P and Q are reciprocal loci, then the curve in the problem *is its own reciprocal*. And we may propose the following question:—

Find the curve or surface, the locus of P, which is its own reciprocal.

We have also analogous to the original problem the following question in Solid Geometry:—

Given the four points A, B, C, D in space, to find the locus of the points P, Q such that

$$PA + AQ = PB + BQ = PC + CQ = PD + DQ = \text{a given line.}]$$

I. *Solution by* F. D. THOMSON, M.A.; *and* E. FITZGERALD.

Taking the three given points A, B, C as the vertices of the triangle of reference, let (α, β, γ) and $(\alpha', \beta', \gamma')$ be the trilinear coordinates of the foci S and H, respectively; then

$$SA^2 = (\beta^2 + \gamma^2 + 2\beta\gamma \cos A) \operatorname{cosec}^2 A;$$

or, using *areal* coordinates for greater convenience, let

$x : a\alpha = y : b\beta = z : c\gamma = 1 : 2\Delta$; then $SA^2 = c^2y^2 + b^2z^2 + 2bcyz \cos A$.

Hence since $SA + HA = SB + HB = SC + HC =$ major axis $= d$ suppose,

$$\sqrt{(c^2y^2 + b^2z^2 + 2bcyz \cos A)} + \sqrt{(c^2y'^2 + b^2z'^2 + 2bcy'z \cos A)} = d,$$

and there are two similar equations. We have, therefore,

$$c^2y^2 + b^2z^2 + 2bcyz \cos A = (d-u)^2 \text{ suppose} = p \text{ suppose} \ldots\ldots\ldots \text{(i.)}$$

$$a^2z^2 + c^2x^2 + 2cazx \cos B = (d-v)^2 \text{ suppose} = q \text{ suppose} \ldots\ldots\ldots \text{(ii.)}$$

$$b^2x^2 + a^2y^2 + 2abxy \cos C = (d-w)^2 \text{ suppose} = r \text{ suppose} \ldots\ldots \text{(iii.)}$$

Now $x + y + z = 1$; hence, substituting for x in (ii.) and (iii.), it will be found that these equations become, respectively,

$$cy + bz \cos A = \frac{p-q+c^2}{2c} = t \text{ suppose}, \quad bz + cy \cos A = \frac{p-r+b^2}{2b} = t' \text{ suppose};$$

therefore $cy \sin^2 A = t - t' \cos A$, and $bz \sin^2 A = t' - t \cos A$;

therefore, substituting these values in (i.), we get

$$p \sin^4 A = (t - t' \cos A)^2 + (t' - t \cos A)^2 + 2(t - t' \cos A)(t' - t \cos A) \cos A,$$

whence $p \sin^2 A = t^2 + t'^2 - 2tt' \cos A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv.)}$

This equation contains only α', β', γ' and constants, and is therefore the locus of H, and consequently also of S. The equation (iv.) contains *six* different radicals, viz., u, v, w, uv, vw, wu, and may be made homogeneous by multiplying the different terms by the proper power of $(x + y + z)$.

II. *Solution by the* PROPOSER.

In general if a, b, c be the sides of a triangle, and f, g, h the distances of any point from the angles of the triangle (or, what is the same thing, if (a, b, c, f, g, h) be the distances of any four points in a plane from each other), then we have a certain relation

$$\phi(a, b, c, f, g, h) = 0.$$

Hence if r, s, t be the distances of the one focus from the angles of the triangle, and the major axis is $= 2\lambda$, then the distances for the other focus are $2\lambda - r$, $2\lambda - s$, $2\lambda - t$; and considering the three angles and the other focus as a system of four points, we have

$$\phi(a, b, c, 2\lambda - r, 2\lambda - s, 2\lambda - t) = 0,$$

which is a relation between the distances r, s, t of the first focus from the angles of the triangle, and which, treating these distances as coordinates (of course in a generalised sense of the term "Coordinate") may be regarded

as the equation of the required locus. It is to be observed, that we have identically

$$\phi(a, b, c, r, s, t) = 0,$$

and the equation may be expressed in the simplified form

$$\phi(a, b, c, 2\lambda - r, 2\lambda - s, 2\lambda - t) - \phi(a, b, c, r, s, t) = 0.$$

To develope the solution, I notice that the expression for the equation $\phi(a, b, c, f, g, h,) = 0$ is

$$\begin{aligned} &b^2c^2(g^2+h^2) + c^2a^2(h^2+f^2) + a^2b^2(f^2+g^2) \\ &+g^2h^2(b^2+c^2) + h^2f^2(c^2+a^2) + f^2g^2(a^2+b^2) \\ &-a^2f^2(a^2+f^2) - b^2g^2(b^2+g^2) - c^2h^2(c^2+h^2) \\ &-a^2g^2h^2 - b^2h^2f^2 - c^2f^2g^2 - a^2b^2c^2 = 0\,; \end{aligned}$$

see my paper, "Note on the value of certain determinants, &c.," *Quarterly Math. Jour.*, Vol. iii. (1860) pp. 275—277. Or, as this may also be written

$$\Sigma\left\{(b^2+c^2-a^2)(g^2h^2+a^2f^2) - a^2f^4\right\} - a^2b^2c^2 = 0,$$

where Σ refers to the simultaneous cyclical permutation of (a, b, c) and of $(f, g, h,)$. Hence we have only in this equation to write $2\lambda - r$, $2\lambda - s$, $2\lambda - t$ in place of (f, g, h), and to omit the terms independent of λ, being in fact those which are equal to $\phi(a, b, c, r, s, t)$. Observing that we have

$$g^2h^2 + a^2f^2 = \left\{4\lambda^2 - 2\lambda(s+t) + st\right\}^2 + a^2(2\lambda - r)^2$$

$$= 16\lambda^4 - 16\lambda^3(s+t) + 4\lambda^2(s^2+t^2+4st+a^2) - 4\lambda\left[st(s+t) + a^2r\right] + s^2t^2 + a^2r^2;$$

$$f^4 = (2\lambda - r)^4 = 16\lambda^4 - 32\lambda^3 r + 24\lambda^2r^2 - 8\lambda r^3 + r^4,$$

the equation becomes

$$\begin{aligned} &16\lambda^4\left\{\Sigma(b^2+c^2-a^2) - \Sigma a^2\right\} \\ &-16\lambda^3\left\{\Sigma(b^2+c^2-a^2)(s+t) - 2\Sigma a^2r\right\} \\ &+4\lambda^2\left\{\Sigma(b^2+c^2-a^2)(s^2+t^2+4st+a^2) - 6\Sigma a^2r^2\right\} \\ &-4\lambda\left\{\Sigma(b^2+c^2-a^2)\left[st(s+t)+a^2r\right] - 2\Sigma a^2r^3\right\} = 0, \end{aligned}$$

where the Σ's refer to the simultaneous cyclical permutation of the (a, b, c) and the (r, s, t). The coefficients of λ^4 and λ^3 are, it is easy to see, each $=0$; and in the coefficient of λ^2 the terms $\Sigma(b^2+c^2-a^2)(s^2+t^2) - 6\Sigma a^2r^2$ are $= -4\Sigma a^2r^2$; hence dividing the whole equation by 4λ, we find

$$\begin{aligned} &\lambda\left\{\Sigma(b^2+c^2-a^2)(4st+a^2) - 4\Sigma a^2r^2\right\} \\ &-\left\{\Sigma(b^2+c^2-a^2)\left[st(s+t)+a^2r\right] - 2\Sigma a^2r^3\right\} = 0, \end{aligned}$$

which is the required relation between (r, s, t).

It may be noticed that, expressing the distances r, s, t in terms of Cartesian or trilinear coordinates (x, y) or (x, y, z), then r^2, s^2, t^2 are rational and integral functions of the coordinates, and the form of the equation therefore is

$$A_2 + B_2r + C_2s + D_2t + E_0st + F_0tr + G_0rs = 0,$$

where the subscript numbers denote the degrees in regard to the coordinates. Multiplying this equation successively by $1, r, s, t, st, tr, rs, rst$, we have eight equations linear in the last mentioned eight quantities, the coefficients being of known degrees respectively; and eliminating the eight quantities,

we have the rationalised equation expressed in the form, determinant (of order 8)$=0$; viz. this is

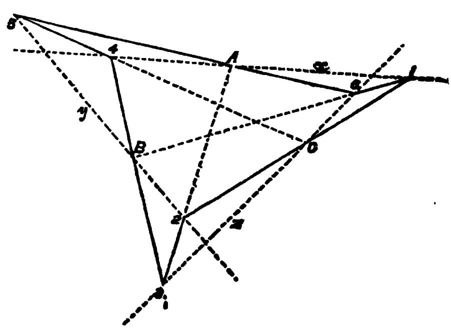
$$\begin{vmatrix} A_2 , & B_2 , & C_2 , & D_2 , & E_0 , & F_0 , & G_0 , & 0 \\ B_2r^2 , & A_2 , & G_0r^2 , & F_0r^2 , & 0 , & D_2 , & C_2 , & E_0 \\ C_2s^2 , & G_0s^2 , & A_2 , & E_0s^2 , & D_2 , & 0 , & B_2 , & F_0 \\ D_2t^2 , & F_0t^2 , & E_0t^2 , & A_2 , & C_2 , & B_2 , & 0 , & G_0 \\ E_0s^2t^2, & 0 , & D_2t^2 , & C_2s^2 , & A_2 , & G_0s^2 , & F_0t^2 , & B_2 \\ F_0t^2r^2, & D_2t^2 , & 0 , & B_2r^2 , & G_0r^2 , & A_2 , & E_0t^2 , & C_2 \\ G_0r^2s^2, & C_2s^2 , & B_2r^2 , & 0 , & F_0r^2 , & E_0s^2 , & A_2 , & D_2 \\ 0 , & E_0s^2t^2, & F_0t^2r^2, & G_0r^2s^2, & B_2r^2 , & C_2s^2 , & D_2t^2 , & A_2 \end{vmatrix} = 0.$$

This seems to be of the degree 18 in the coordinates, but it is probable that the real degree is lower.

1652. (Proposed by W. K. CLIFFORD.)—Through the angles A, B, C of a plane triangle three straight lines Aa, Bb, Cc are drawn. A straight line AR meets Cc in R; RB meets Aa in P; PC meets Bb in Q; QA meets Cc in r; and so on. Prove that, after going twice round the triangle in this way, we always come back to the same point.

Show that the theorem is its own reciprocal. Find the analogous properties of a skew quadrilateral in space, and of a polygon of n sides in a plane.

Solution by PROFESSOR CAYLEY.



1. The theorem may be thus stated: Given three lines x, y, z, and in these lines respectively the points A, B, C; then there exist an infinity of hexagons, such that the pairs of opposite angles lie in the lines x, y, z, respectively, and that the pairs of opposite sides pass through the points A, B, C, respectively.

2. The demonstration is as follows:—We have to show that, starting from an arbitrary point 1 in the line x, and constructing in the prescribed manner (as shown successively in the figure) the points 2, 3, 4, 5, 6, the last side 61 of the hexagon 123456 will pass through B. By the construction, we have A, 2, 3 in a line, and likewise C, 4, 5; hence, by Pascal's theorem, applied to the six points in a pair of lines, the points of intersection of the lines (25, 34), (3C, A5), (A4, C2), that is, the points B, 6, 1, lie in a line; which is the required theorem.

3. More generally suppose that the points A, B, C are not on the lines x, y, z, respectively. I remark that it is not in general possible to describe a hexagon such that the opposite angles lie in the lines x, y, z, respectively, and the opposite sides pass through the points A, B, C, respectively; but if there exists one hexagon (viz., a proper hexagon, not a triangle twice repeated), then there exists an infinity of such hexagons.

4. In fact, if it be required to find a polygon, the angles whereof lie in given lines respectively, and the sides whereof pass through given points respectively; the problem is either indeterminate or admits of only *two* solutions. If therefore in any particular case there are three or more solutions, the problem is indeterminate, and has an infinity of solutions. Now, in the above mentioned case of the three lines and the three points, there exist *two* triangles, the angles whereof lie in the given lines, and the sides pass through the given points; and each triangle, taking the angles twice over in the same order 123123, is a hexagon satisfying the conditions of the problem; hence, if we have besides a proper hexagon satisfying the conditions of the problem, there are really *three* solutions, and the problem is therefore indeterminate.

5. Suppose that the three lines x, y, z, and also two of the three points, say the points A and B, are given; we may construct geometrically a locus, such that, taking for C any point of this locus, the problem shall be indeterminate: in fact, starting with the point 4, and constructing successively the points 3, 2; taking an arbitrary direction for the line 21, and constructing successively the points 1, 6, 5; then the intersection of the lines 21 and 54 is a position of the point C: and by taking any number of directions for the line 21, we obtain for each of them a different position of the point C; and so construct the locus.

6. The locus in question is, as will be shown, a line; and if the point A is on the line x, and the point B on the line y, then the locus of C will be the line z; that is, C being any point of the line z, the problem is indeterminate; which is Mr. Clifford's theorem.

7. To prove this, consider the lines x, y, z, and also the points A, B, C, as given; the point 1 is an arbitrary point on the line x, linearly determined by means of a parameter u; and for every position of the point 1 we have a corresponding position of the point 4; let u' be the corresponding parameter for the point 4; the series of points 1 is homographic with the series of points 4; that is, the parameters u, u' are connected by an equation of the form $auu' + bu + cu + d = 0$, (where of course a, b, c, d are functions of the parameters which determine the given lines x, y, z and points A, B, C.) But if the problem be indeterminate, then starting from the point 1 and constructing the point 4, and again starting from the point 4 and making the very same construction, we arrive at the original point 1, that is, u must be the same function of u' that u' is of u; and this will be the case if $b=c$; hence $b=c$ is the condition in order that the problem may be indeterminate.

8. To effect the calculation, take $x=0$, $y=0$, $z=0$ for the equations of the lines x, y, z, respectively; and let (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ be the coordinates of the points A, B, C respectively. Let 1 and 4 be given as the intersections of the line $x = 0$ with the lines $y-uz = 0$, $y-u'z = 0$, respectively; and assume that for the point 2 we have $y = 0$, $z - vx = 0$, and for the point 3, $z=0$, $x-wy=0$. Then 1, C, 2 are in a line; as are also 2, A, 3; 3, B, 4; hence we obtain

$$v = \frac{\gamma''u-\beta''}{\alpha''u}, \quad w = \frac{\alpha v-\gamma}{\beta v}, \quad u' = \frac{\beta' w-\alpha}{\gamma' w};$$

therefore, eliminating v and w, we have

$$(\alpha\gamma''-\alpha''\gamma)\,\gamma' uu'-\alpha\beta''\gamma' u'-(\alpha\beta'\gamma''-\alpha'\beta\gamma''-\alpha''\beta'\gamma)\,u-\beta''(\alpha'\beta-\alpha\beta)=0.$$

The required condition, therefore, is

$$\alpha\beta''\gamma'=\alpha\beta'\gamma''-\alpha'\beta\gamma''-\alpha''\beta'\gamma, \quad \text{or} \quad \alpha\beta'\gamma''-\alpha\beta''\gamma'-\alpha'\beta\gamma''-\alpha''\beta'\gamma=0\,;$$

which is linear in regard to each of the three sets (α,β,γ), (α',β',γ'), $(\alpha'',\beta',\gamma'')$, separately; that is, two of the points A, B, C being given, the locus of the remaining point is a line. In particular, if $\alpha=0$, $\beta'=0$; then the equation becomes $\alpha'\beta\gamma''=0$, and assuming that neither $\alpha'=0$ or $\beta=0$, then the equation becomes $\gamma''=0$, that is, A, B being arbitrary points on the lines $x=0$, $y=0$ respectively, the locus of C is the line $z=0$.

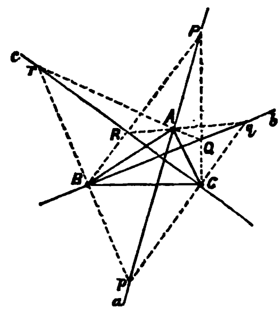
9. Mr. Clifford's theorem is clearly its own reciprocal. I do not know the *precise* analogues of his special form of the theorem; but the analogue of the more general theorem stated in (6) is as follows: viz., we may have in the plane n lines $x, y, z \ldots$ and n points A, B, C . . ., such that there exist an infinity of $2n$-gons whereof the pairs of opposite angles lie in the given lines respectively; and the pairs of opposite sides pass through the given points respectively; and if the n lines and $n-1$ of the n points be assumed at pleasure, then the locus of the remaining point is a line. It is moreover clear by the principle of reciprocity, that if the n points and $n-1$ of the n lines be assumed at pleasure, then the envelope of the remaining line is a point.

There exists also an analogue in space; viz.,—we may have n lines x, y, z, . . and n lines A, B, C . . . such that there exist an infinity of (skew) $2n$-gons whereof the pairs of opposite angles lie in the given lines x, y, z . . respectively; and the pairs of opposite sides meet in the given lines A, B, C, . . . respectively. It may be added, that if all but one of the $2n$ lines $x, y, z \ldots$ A, B, C . . . are given, then the 'six coordinates' of the remaining line satisfy a certain linear equation, but I do not stop to explain the geometrical interpretation of this theorem.

10. Referring to the foregoing figure, if instead of the point 1 we take on the line x, a point $1'$, and construct therewith the hexagon $1'2'3'4'5'6'$; then if α, α' be the (foci or) sibi-conjugate points of the range $1, 4, 1', 4'$ on the line x; β, β' the sibi-conjugate points of the range $2, 5, 2', 5'$ on the line y; and γ, γ' the sibi-conjugate points of the range $3, 6, 3', 6'$ on the line z;—the points in question form two triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, such that for each triangle the angles lie in the given lines and the sides pass through the given points. This is an elegant geometrical construction for the problem of the in-and-circumscribed triangle, in the particular case where the given points A, B, C lie in the given lines x, y, z, respectively.

11. The points 1, 2, 3, 4, 5, 6, A, B, C constitute a system of 9 points which lie in 9 lines of 3 each. The points $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, A, B, C constitute a radically distinct system of 9 points lying in 9 lines of 3 each; viz., in the former system there are 3 sets of 3 lines which contain all the 9 points; in the latter system there is only the set of lines $A\alpha\alpha'$, $B\beta\beta'$, $C\gamma\gamma'$ which contains all the 9 points. The last-mentioned system may be constructed as follows:—The points β, β' and γ, γ' are arbitrary: A is the intersection of the lines $\beta\gamma$ and $\beta'\gamma'$; and then joining A with the point of intersection of the lines $\beta\gamma'$ and $\beta'\gamma$ we have α an arbitrary point on the joining line; the lines $\alpha\gamma$ and $\beta\beta'$ meet in the point B, the lines $\alpha\beta$ and $\gamma\gamma'$ in the point C; the lines $C\beta'$ and $B\gamma'$ will then meet in a point α' on the line $A\alpha$; and we have thus the figure of the nine points $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, A, B, C.

II. *Solution by the* PROPOSER; J. DALE; E. FITZGERALD; REV. R. TOWNSEND, M.A.; *and others.*



Let x, y, z be the sides of the triangle ABC, and let $ay=z$, $bz=x$, $cx=y$ be the three lines drawn through them. Start with the line AR or $y=z$, which meets Cc or $cx=y$ on $cx=z$, which meets $ay=z$ on $cx=ay$, which meets $bz=x$ on $cbz=ay$, which meets $cx=y$ on $bz=ax$, which meets $ay=z$ on $by=x$, which meets $bz=x$ on $y=z$; so that we have come round again. The extension of this is now easy; I write down two enunciations :—

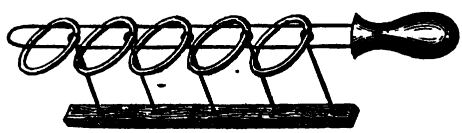
Consider a plane polygon of an odd number of sides; let the two sides adjacent to any given side be produced to meet, and through their intersections let an arbitrary line be drawn; then treating these lines in the same way as Aa, Bb, Cc, were treated in the case of the triangle, we may go twice round the polygon, and shall always come back to the same point.

Let ABCD be a skew quadrilateral in space, and through the four sides AB, BC, CD, DA let arbitrary planes be drawn; let any line through A meet the plane through CD in a; aB meets the plane DA in b; and so on; after going *three* times round the quadrilateral we shall come back to the same point.

The theorem is not true for a plane polygon of an even number of sides; I have not been able to find an analogue in this case.

1632. (Proposed by H. J. PURKISS, B.A.)—If there be n rings in a system of *complicati annuli* (the common ring-puzzle) determine the number of operations required to play them all off the bow.

Solution by the PROPOSER.



The system referred to (which may be found in most toy-shops) is that represented in the figure. The wire from which the rings are to be played off is usually called the "bow." Each pin passes through the ring behind it, and holds its own ring. The puzzle is to remove the rings from the bow.

Now in order to play off the nth ring we must previously remove the first

$(n-2)$; we can then remove the nth. In order to remove the $(n-1)$th we have to put the first $(n-2)$ on again. Hence if u_n be the number of operations required, we have

$$u_n = u_{n-2}+1+u_{n-2}+u_{n-1}, \text{ or } u_n-u_{n-1}-2u_{n-2}=1.$$

The solution of this equation is $u_n = A2^n + B(-1)^n - \frac{1}{2}$; hence, observing that $u_n=1$ when n is 1 or 2, we have

$$u_n = 2^{n-1} - \tfrac{1}{2}\left\{1+(-1)^n\right\}.$$

1653. (Proposed by Dr. Booth, F.R.S.)—Let t be the distance between the point of contact of any tangent plane to a surface and the foot of the perpendicular drawn on it from the origin of coordinates, and let $\Phi \equiv \phi(\xi, v, \zeta) = 0$ be the tangential equation of the surface; then

$$t^2 = \frac{\left(\frac{d\Phi}{d\xi}v - \frac{d\Phi}{dv}\xi\right)^2 + \left(\frac{d\Phi}{dv}\zeta - \frac{d\Phi}{d\zeta}v\right)^2 + \left(\frac{d\Phi}{d\zeta}\xi - \frac{d\Phi}{d\xi}\zeta\right)^2}{\left\{\frac{d\Phi}{d\xi}\xi + \frac{d\Phi}{dv}v + \frac{d\Phi}{d\zeta}\zeta\right\}^2(\xi^2+v^2+\zeta^2)}.$$

Solution by the Proposer; E. Fitzgerald; J. Dale; X. U. J.; *and others.*

Let r be the radius vector of the point of contact, and p the perpendicular from the origin on the tangent plane; then we shall have

$$t^2 = r^2 - p^2 = (x^2+y^2+z^2)-(\xi^2+v^2+\zeta^2)^{-1};$$

and in the Solution of Question 1509 (see *Reprint*, Vol. II. p. 20) it is shown that

$$x = \frac{d\Phi}{d\xi} + \left\{\frac{d\Phi}{d\xi}\xi + \frac{d\phi}{dv}v + \frac{d\Phi}{d\zeta}\zeta\right\},$$ with similar expressions for y and z;

hence, substituting these values of x, y, z in the preceding equation, we get, after some obvious reductions, the above general expression for t.

1655. (Proposed by M. W. Crofton, B.A.)—Let the equations of two circles whose radii are r, r' be denoted by $\Theta = 0$, $\Theta' = 0$; then the two circles whose equations are

$$\frac{\Theta}{r} + \frac{\Theta'}{r'} = 0, \quad \frac{\Theta}{r} - \frac{\Theta'}{r'} = 0$$

intersect at right angles.

I. *Solution by the* REV. R. TOWNSEND, M.A.

The two circles in question are evidently the two, coaxal with Θ and Θ', loci of points the squares on the tangents from which to Θ and Θ' have the ratio of their radii $r : r'$; they have therefore their centres at the two centres of perspective of Θ and Θ' (Townsend's *Modern Geometry*, Vol. I. Art. 192. Cor. 1°), and consequently bisect internally and externally the angles of intersection of Θ and Θ'; hence they intersect at right angles.

II. *Solution by* E. FITZGERALD; *and others.*

If (ξ, η) be the coordinates of one of the common points of the four circles, the equations of the tangents at (ξ, η) to the two circles in question will be

$$\left(\frac{1}{r}\frac{d\Theta}{d\xi}+\frac{1}{r'}\frac{d\Theta'}{d\xi}\right)(x-\xi)+\left(\frac{1}{r}\frac{d\Theta}{d\eta}+\frac{1}{r'}\frac{d\Theta'}{d\eta}\right)(y-\eta)=0 \quad\ldots\ldots\ldots\ldots(\alpha),$$

$$\left(\frac{1}{r}\frac{d\Theta}{d\xi}-\frac{1}{r'}\frac{d\Theta'}{d\xi}\right)(x-\xi)+\left(\frac{1}{r}\frac{d\Theta}{d\eta}-\frac{1}{r'}\frac{d\Theta'}{d\eta}\right)(y-\eta)=0 \quad\ldots\ldots\ldots\ldots(\beta).$$

Also, since the arcs of different circles are proportional to their radii, if s, s' be arcs of the circles Θ, Θ' respectively, we have

$$r^2 : r'^2 = ds^2 : ds'^2 = \left(\frac{d\Theta}{d\xi}\right)^2+\left(\frac{d\Theta}{d\eta}\right)^2 : \left(\frac{d\Theta'}{d\xi}\right)^2+\left(\frac{d\Theta'}{d\eta}\right)^2,$$

whence $$\left\{\frac{1}{r^2}\left(\frac{d\Theta}{d\xi}\right)^2-\frac{1}{r'^2}\left(\frac{d\Theta'}{d\xi}\right)^2\right\}+\left\{\frac{1}{r^2}\left(\frac{d\Theta}{d\eta}\right)^2-\frac{1}{r'^2}\left(\frac{d\Theta'}{d\eta}\right)^2\right\}=0.$$

Since, therefore, the product of the coefficients of x in the equations (α), (β) added to the product of the coefficients of y in the same equations, is equal to zero, it follows that these two lines, and therefore also the circles to which they are tangents, intersect at right-angles.

III. *Solution by* F. D. THOMSON, M.A.; J. DALE; *and others.*

Let $\Theta \equiv x^2+y^2-2ax-2by+a^2+b^2-r^2=0$ be equation to 1st circle,

and $\Theta' \equiv x^2+y^2-2a'x-2b'y+a'^2+b'^2-r'^2=0$ be equation to 2nd circle;

then $$\frac{\Theta}{r}+\frac{\Theta}{r} \equiv \left(\frac{1}{r}+\frac{1}{r'}\right)(x^2+y^2)-2\left(\frac{a}{r}+\frac{a'}{r'}\right)x-2\left(\frac{b}{r}+\frac{b'}{r'}\right)y$$
$$+\frac{a^2+b^2}{r}+\frac{a'^2+b'^2}{r'}-(r+r')=0 \quad\ldots\ldots\ldots\ldots(\text{i}),$$

and $$\frac{\Theta}{r}-\frac{\Theta}{r'} \equiv \left(\frac{1}{r}-\frac{1}{r'}\right)(x^2+y^2)-2\left(\frac{a}{r}-\frac{a'}{r'}\right)x-2\left(\frac{b}{r}-\frac{b'}{r'}\right)y$$
$$+\frac{a^2+b^2}{r}-\frac{a'^2+b'^2}{r'}-(r-r')=0 \quad\ldots\ldots\ldots\ldots(\text{ii}).$$

Let (A, B, R), (A′, B′, R′) be coordinates of centre and radii of (i), (ii.); then

$$A = \frac{a'r+ar'}{r+r'},\quad B = \frac{b'r+br'}{r+r'},\quad A' = \frac{ar'-a'r}{r'-r},\quad B' = \frac{br'-b'r}{r'-r};$$

$$\therefore\ A-A' = \frac{2(a'-a)\,rr'}{r'^2-r^2};\ \text{and similarly } B-B' = \frac{2(b'-b)\,rr'}{r'^2-r^2},$$

$$\therefore\ (A-A')^2+(B-B')^2 = \frac{4r^2r'^2}{(r'^2-r^2)^2}\left\{(a'-a)^2+(b'-b)^2\right\} \ldots\ldots\ldots\ \text{(iii)}.$$

$$\text{Now } R^2 = -\frac{(a^2+b^2)\,r'+(a'^2+b'^2)\,r}{r+r'}+rr'+(A^2+B^2),$$

$$\text{and}\qquad A^2+B^2 = \frac{(a^2+b^2)\,r'^2+(a'^2+b'^2)\,r^2+2\,(aa'+bb')\,rr'}{(r+r')^2},$$

$$\text{therefore}\qquad R^2 = -\frac{\left\{(a-a')^2+(b-b')^2\right\}rr'}{(r+r')^2}+rr';$$

$$\text{similarly}\qquad R'^2 = \frac{\left\{(a-a')^2+(b-b')^2\right\}rr'}{(r-r')^2}-rr';$$

$$\text{therefore}\qquad R^2+R'^2 = \frac{4r^2r'^2\left\{(a-a')^2+(b-b')^2\right\}}{(r^2-r'^2)^2}\ \ldots\ldots\ldots\ \text{(iv.)}$$

Hence $R^2+R'^2 =$ (iii.) = square of the distance between the centres, and therefore the circles (i.) and (ii.) cut each other at right angles.

1656. (Proposed by E. McCormick.)—If a circle touch an ellipse and its two directrices in four points, prove that its centre is at the end of the minor axis.

Solution by J. Taylor; R. Tucker, M.A.; E. Fitzgerald; J. Dale; *and others.*

It is easy to see that the centre of the circle must be *somewhere* in the minor axis of the ellipse; therefore its radius (r) is equal to ae^{-1}. The equations of the ellipse and circle, referred to the centre of the ellipse as origin and its axes as axes of coordinates, are

$$b^2x^2+a^2y^2=a^2b^2,\quad x^2+(y-k)^2=r^2.$$

Eliminating x^2, we have $(a^2-b^2)\,y^2+2b^2ky+b^2\,(r^2-a^2-k^2)=0.$

The condition that the curves should touch one another is

$b^2k^2=(a^2-b^2)\,(r^2-a^2-k^2)$, or $a^2k^2=(a^2-b^2)\,(r^2-a^2)=a^2b^2$; $\therefore\ k=\pm b.$

Hence the centre of the circle is at the extremity of the minor axis.

[NOTE.—The coordinates of the points of contact of the circle and the ellipse are readily found to be

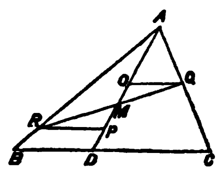
$$x = \pm \frac{a}{e^2}(2e^2-1)^{\frac{1}{2}}, \quad y = \pm \frac{a}{e^2}(1-e^2)^{\frac{3}{2}}:$$

whence it is obvious that the two points of contact coincide at the other end of the minor axis when $e^2 = \frac{1}{2}$, and when $e^2 < \frac{1}{2}$ the circle is imaginary.—EDITOR.]

1664. (Proposed by D. M. ANDERSON.)—Q is any point in the side AC of a triangle ABC, R any point in AB, M the middle point of the line QR, and D the point in which the line AM meets BC. Prove that

$$BD : DC = \frac{BA}{AC} : \frac{RA}{AQ}.$$

Solution by the PROPOSER; ALPHA; J. DALE; E. FITZGERALD; R. TUCKER, M.A.; J. TAYLOR; *and others.*



Draw QO and RP parallel to BC; then we have QM = MR, and QO = RP; also BD : RP = BA : RA, and QO : DC = AQ : AC; therefore

$$BD : DC = BA \,.\, AQ : RA \,.\, AC = \frac{BA}{AC} : \frac{RA}{AQ}.$$

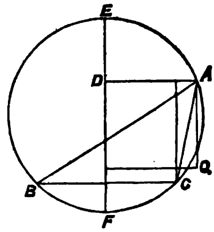
From this we readily derive, by Ceva's theorem, the following property, which includes that in Question 1616 as a particular case.

If through any point within a triangle $O_1O_2O_3$, straight lines O_1A, O_2B, O_3C be drawn from the vertices to meet the opposite sides in A, B, C; and if K_1, K_2, K_3 be the middle points of BC, CA, AB respectively; then O_1K_1, O_2K_2, O_3K_3 will be concurrent.

If, in the above figure, the points B, R, Q, C lie in a circle, or, what amounts to the same thing, if the triangle AQR is similar to ABC (the angle Q being equal to B, and R to C), then AQ : RA = BA : AC, and therefore $BD : DC = BA^2 : AC^2$; that is to say, BC is divided at D into segments which have to one another the duplicate ratio of the adjacent sides of the triangle. This suggests a simple construction for dividing a given line into segments which shall have to one another the duplicate ratio of two given lines.

1631. (Proposed by J. O'CALLAGHAN.)—In a given circle to inscribe a triangle such that its vertex shall be at a fixed point on the circumference, its base parallel to a line given in position, and its area given or a maximum.

Solution by the PROPOSER; ALPHA; *and others.*

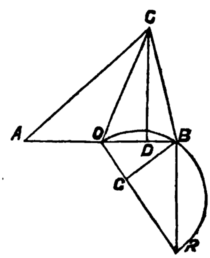


Let A be the fixed point on the given circle EAF, AD parallel to the line given in position. Draw the diameter EF perpendicular to AD; make the rectangle DQ equal to the *given area;* and through Q, along the asymptotes DA, DF, draw an equilateral hyperbola cutting the given circle in C; then, if CB be drawn perpendicular to EF, ABC will be the triangle required; since $\triangle$ ABC = DC = DQ = the *given area.*

The triangle ABC will be a *maximum* when the hyperbola *touches* the circle, that is, when C is such a point that the portion of the common tangent through C, limited by DA, DF, is *bisected* at the point of contact.

1642. (Proposed by P. W. FLOOD.)—Given the sum of the squares on the sides containing the vertical angle, and the difference of the segments of the base made by the perpendicular; to construct the triangle when the solid contained under the base and square on the perpendicular is a maximum.

Solution by J. O'CALLAGHAN; ALPHA; *the* PROPOSER; *and others.*



Take a line OR such that the square thereon, together with the square on half the given difference of the segments, shall be equal to the given sum of the squares on the sides. Take $OG = \frac{1}{3}OR$, and draw GB perpendicular to OR, meeting a semicircle on OR in B. Produce BO; and on it make OA = OB, and OD = half the given difference of segments; join OB, BR, and draw DC perpendicular to AB and equal to BR; then ABC shall be the triangle required.

For $\quad \frac{1}{2}(AC^2 + CB^2) = OB^2 + OC^2$

$\quad = OB^2 + OD^2 + BR^2 = OR^2 + OD^2;$

therefore $\quad AC^2 + CB^2 =$ the given sum;

also $\quad AD - DB = 2OD =$ the given difference;

and this is true whatever point G is in OR.

Moreover, since $RG = 2OG$, $OG \cdot GR^2$ is a *maximum* (Simpson's *Geometry*, p. 208); therefore $2OB \cdot BR^2$ $(= AB \cdot CD^2)$ is a maximum.

Hence ABC is the required triangle.

1644. (Proposed by R. TUCKER, M.A.)—Two parabolas, whose parameters are as 8 : 9, have a common vertex and coincident axes; if from any

point on the outer curve two tangents be drawn to the inner curve, show that (1) the tangent of the inclination of one of these lines to the axis is twice that of the other; and (2) the part of either tangent intercepted between the outer curve and the axis is equal to the part within the curve.

Solution by the PROPOSER; R. KNOWLES; *and others.*

1. Generally, suppose the tangent of the inclination of one of the lines to be n times that of the other, and let

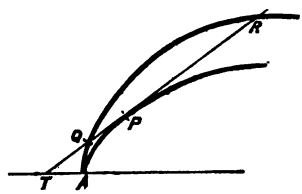
$$y = mx + \frac{a}{m} \ldots\ldots (\alpha), \quad y = mnx + \frac{a}{mn} \ldots\ldots (\beta),$$

be tangents from the same point to the parabola $y^2 = 4ax$; then eliminating between (α) and (β), we get $m^2nx = a$; hence from (α)

$$y^2 = \frac{(n+1)^2}{n} ax. \quad \text{Make } n = 2; \text{ then } y^2 = \frac{9}{8}(4ax);$$

hence the truth of (1) may be inferred.

2. Let RPQ be the tangent to the inner curve, cutting the common axis in T. Take the general case of RT = n . QT, and let (x', y'), (x_1, y_1), (x_2, y_2) be the points P, R, Q respectively; and $y^2 = 4ax$, $y^2 = 4bx$, the respective equations to the inner and outer curves. Then we have



$$y_1 = ny_2 \ldots\ldots (1); \quad y_1y' = 2a(x + x'), \; y_2y' = 2a(x + x') \ldots\ldots (2);$$

and because AT $= x'$, therefore $\dfrac{y_1}{x_1 + x'} = \dfrac{y_2}{x_2 + x'}$ .... (3);

also $y_1^2 = 4bx_1$, $y_2^2 = 4bx_2$; therefore, by (1), $x_1 = n^2x_2$ ...... (4).

From (1), (3), (4), we get $x' = nx_2$; hence from (2), squaring, we have

$$\frac{b}{a} = \frac{(n+1)^2}{4n}. \quad \text{Make } n = 2; \text{ then } \frac{4b}{4a} = \frac{9}{8};$$

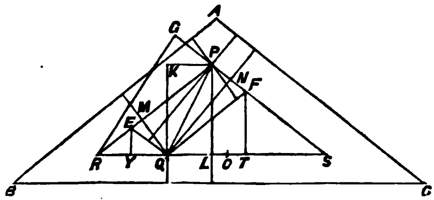
hence the truth of (2) may be inferred.

1657. (Proposed by A. RENSHAW.)—From a point Q in the side RS of a triangle PRS, QE and QF are drawn parallel to PS, PR; also PL, EY, FT are drawn perpendicular to RS, and RG perpendicular to SP; prove that

$$PQ^2 = RP \, . \, PE + SP \, . \, PF - RQ \, . \, QS = RQ \, . \, QY + SQ \, . \, QT - SP \, . \, PG,$$

and deduce therefrom the ordinary expressions for the distance between two points whose trilinear coordinates are given.

Solution by the PROPOSER; ALPHA; J. DALE; E. FITZGERALD; *and others.*



The relation $PQ^2 = RP \, . \, PE + SP \, . \, PF - RQ \, . \, QS$ forms the *first* of Matthew Stewart's *General Theorems;* and a proof, slightly different from Stewart's, together with the application of the theorem to the investigation of an elegant problem in loci, may be seen in Mr. McDowell's Solution of Quest. 1276 of the *Educational Times.* From the foregoing relation we have

$$2PQ^2 = (RP^2+PE^2-RE^2)+(SP^2+PF^2-SF^2)-(RS^2-RQ^2-QS^2)$$
$$= (RQ^2+QE^2-RE^2)+(QS^2+QF^2-SF^2)-(RS^2-RP^2-SP^2),$$

therefore $\qquad PQ^2 = RQ \, . \, QY + SQ \, . \, QT - SP \, . \, PG.$

It is easy to see what alterations of sign are necessary when Q does not lie *between* R and S.

Now suppose the sides of the triangle PRS to be parallel to those of the triangle ABC, in reference to which the trilinear coordinates of Q, P are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$; then, drawing QK, QM, QN, PK perpendicular, respectively, to SR, RP, PS, QK, we obtain, from the *two* foregoing relations,

$$PQ^2 = \frac{QK \, . \, QN}{\sin B \sin A} + \frac{QK \, . \, QM}{\sin A \sin C} - \frac{QM \, . \, QN}{\sin C \sin B}$$
$$= \frac{QM^2 \cos C}{\sin B \sin A} + \frac{QN^2 \cos B}{\sin A \sin C} + \frac{QK^2 \cos A}{\sin C \sin B}.$$

Now $4\Delta^2 = abc(a \sin B \sin C) = \&c.$; hence the expressions become

$$PQ^2 = -\frac{abc}{4\Delta^2}\left\{ a(\beta_1-\beta_2)(\gamma_1-\gamma_2) + b(\gamma_1-\gamma_2)(\alpha_1-\alpha_2) + c(\alpha_1-\alpha_2)(\beta_1-\beta_2) \right\}$$
$$= \frac{abc}{4\Delta^2}\left\{ a \cos A \, (\alpha_1-\alpha_2)^2 + b \cos B \, (\beta_1-\beta_2)^2 + c \cos C \, (\gamma_1-\gamma_2)^2 \right\}.$$

COR. 1.—When RPS is a right angle, $PQ^2 = RQ \, . \, QY + SQ \, . \, QT.$

COR. 2.—Let O be the middle point of RS; then

$$PS^2 - PQ^2 = SL^2 - LQ^2 = QS \, . \, QR + 2QS \, . \, LO,$$

therefore, by the first of the above relations, we have

$$SP \, . \, SF = RP \, . \, PE + 2QS \, . \, LO.$$

1594. (Proposed by the Rev. J. BLISSARD.)—Prove that the coefficient of x^n in the expansion of $e^{-x} \cos \sqrt{(2x-x^2)}$ is

$$\frac{(-)^n 2^{n-1}}{1 \, . \, 2 \, . \, . \, . \, n}\left\{ 2 + \frac{1}{2} \, . \, \frac{n-1}{n+1} + \frac{1}{2 \, . \, 3} \, . \, \frac{(n-1)(n-2)}{(n+1)(n+2)} + \&c. \right\}$$

Solution by the PROPOSER.

Using Representative Notation, assume $(1+x)^{P} = e^{x}$. Differentiate n times with respect to x, and multiply by $(1+x)^{n}$; then we have

$P(P-1)\,..\,(P-n+1)(1+x)^{P} = (1+x)^{n}e^{x}$; or, putting $\dfrac{-x}{1+x}$ for x,

$$P(P-1)\,\ldots(P-n+1)(1+x)^{-P} = (1+x)^{-n}e^{-\frac{x}{1+x}} = \frac{1}{(1+x)^{n}} - \frac{x}{(1+x)^{n+1}} + \frac{1}{(1+x)^{n+2}}\cdot\frac{x^{2}}{1.2} - \&c.;$$

hence, equating coefficients of x^{n}, we have

$$\frac{P^{2}(P^{2}-1^{2})\ldots\ldots\{P^{2}-(n-1)^{2}\}}{1.2\ldots\ldots n} = \frac{n(n+1)\ldots(2n-1)}{1.2\ldots n}$$
$$+\frac{(n+1)(n+2)\ldots(2n-1)}{1.2\ldots(n-1)} + \frac{1}{1.2}\cdot\frac{(n+2)(n+3)\ldots(2n-1)}{1.2\ldots(n-2)} + \&c.;$$

therefore $\dfrac{P^{2}(P^{2}-1^{2})\ldots\{P^{2}-(n-1)^{2}\}}{1.2\ldots 2n} =$

$$\frac{1}{2}\cdot\frac{1}{1.2\ldots n}\left\{2 + \frac{1}{1.2}\cdot\frac{n-1}{n+1} + \frac{1}{1.2.3}\cdot\frac{(n-1)(n-2)}{(n+1)(n+2)} + \&c.\right\}.$$

Again, in $(1+x)^{P} = e^{x}$, let $1+x = e^{\theta}$; then we have

$$e^{P\theta} = e^{e^{\theta}-1} = \frac{1}{e}\cdot e^{e^{\theta}};$$

$$\therefore\quad 2\cos P\theta = \frac{1}{e}\left(e^{e^{i\theta}} + e^{e^{-i\theta}}\right) = \frac{1}{e}\left(e^{\cos\theta + i\sin\theta} + e^{\cos\theta - i\sin\theta}\right)$$
$$= \frac{2}{e}\cdot e^{\cos\theta}\cdot\cos(\sin\theta).$$

therefore $\qquad \cos 2P\theta = \dfrac{1}{e}\cdot e^{\cos 2\theta}\cdot\cos(\sin 2\theta).$

Now, in the formula $\cos n\theta = 1 - \dfrac{n^{2}}{1.2}\sin^{2}\theta + \dfrac{n^{2}(n^{2}-2^{2})}{1.2.3.4}\sin^{4}\theta - \&c.$, put $2P$ for n; then we have

$$\cos 2P\theta = 1 - \frac{P^{2}}{1.2}(2\sin\theta)^{2} + \frac{P^{2}(P^{2}-1^{2})}{1.2.3.4}(2\sin\theta)^{4} - \&c.$$
$$= \frac{1}{e}e^{\cos 2\theta}\cdot\cos(\sin 2\theta).$$

Let $2\sin\theta = \sqrt{(2x)}$, then $\cos 2\theta = 1-x$, and $\sin 2\theta = \sqrt{(2x-x^{2})}$;

$$\therefore\quad 1 - \frac{P^{2}}{1.2}(2x) + \frac{P^{2}(P^{2}-1^{2})}{1.2.3.4}(2x)^{2} - \&c. = e^{-x}\cdot\cos\sqrt{(2x-x^{2})};$$

hence, equating coefficients of x^{n}, we have

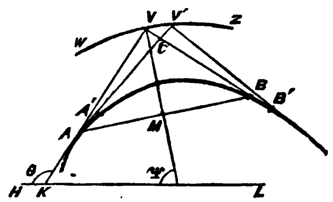
$$C_{n}\left\{e^{-x}\cos\sqrt{(2x-x^{2})}\right\} = (-)^{n}2^{n}\frac{P^{2}(P^{2}-1^{2})\ldots\ldots\{P^{2}-(n-1)^{2}\}}{1.2\ldots 2n}$$
$$= \frac{(-)^{n}2^{n-1}}{1.2\ldots n}\left\{2 + \frac{1}{2}\cdot\frac{n-1}{n+1} + \frac{1}{2.3}\cdot\frac{(n-1)(n-2)}{(n+1)(n+2)} + \&c.\right\}.$$

1622. (Proposed by M. W. CROFTON, B.A.)—Let A, B be any two points on any plane curve, the tangents at which AV, BV meet at right angles: prove that the normal at V to the curve which is the locus of V bisects the chord AB; and that its radius of curvature there is

$$R = \frac{(T^2+T'^2)^{\frac{3}{2}}}{2(T^2+T'^2)-T\rho'-T'\rho},$$

where $T=AV$, $T'=BV$, and ρ, ρ' are the radii of curvature at A, B.

Solution by the PROPOSER; *and* F. D. THOMSON, M.A.



Let V′ be the consecutive point of the locus WZ of V; and put $\theta = VKH$, the inclination of T to a fixed axis. It is easily seen, by the infinitesimal method, that the curve VV′ touches the semicircle on AB; hence its normal VM bisects AB. Now $Vt = Td\theta$, $V't = T'd\theta$, the angle VTV′ being a right angle in the limit; hence, putting $d\Sigma = VV'$, $ds = AA'$, $ds' = BB'$, we have

$$VV'^2 = Vt^2 + V't^2, \quad \text{or} \quad d\Sigma^2 = (T^2+T'^2)\,d\theta^2 \;\ldots\ldots\ldots\ldots (1).$$

Again, let ψ be the inclination of VM to the fixed axis L;

then $$\theta - \psi = AVM = VAM = \tan^{-1}\frac{T'}{T};$$

hence, R being the radius of curvature at V, we have

$$R = \frac{d\Sigma}{d\psi} = \frac{(T^2+T'^2)^{\frac{1}{2}}\,d\theta}{d\theta - d\,.\tan^{-1}\dfrac{T'}{T}} = \frac{(T^2+T'^2)^{\frac{3}{2}}}{T^2+T'^2+\dfrac{T'dT-TdT'}{d\theta}} \;\ldots\ldots\ldots\ldots (2).$$

But $dT = V't - AA' = T'd\theta - ds$, and $dT' = BB' - Vt = ds' - Td\theta$,

or $$\frac{dT}{d\theta} = T' - \rho, \quad \frac{dT'}{d\theta} = \rho' - T \;\ldots\ldots\ldots\ldots\ldots (3);$$

hence, substituting these values in (2), we obtain the proposed expression.

If the curve AB be given by an *intrinsic* equation $\rho = f(\theta)$, the values of T, T′ may be found by integrating the following equations, derived from (3);

$$\frac{d^2T}{d\theta^2} + T = f(\theta + \tfrac{1}{2}\pi) - f'(\theta), \quad \frac{d^2T'}{d\theta^2} + T' = f(\theta) + f'(\theta + \tfrac{1}{2}\pi);$$

and T, T′ being thus known as functions of θ, the intrinsic equation of the locus of V may be found by eliminating θ from the two equations

$$\Sigma = \int (T^2+T'^2)^{\frac{1}{2}}\,d\theta, \quad \psi = \theta - \tan^{-1}\frac{T'}{T}.$$

We thus get a relation between the arc Σ of the new curve and the inclination ψ of the normal to a fixed axis. This method may be applied without difficulty to the cycloid and some other curves.

1630. (Proposed by Dr. SALMON, F.R.S.)—A man has drawn balls from an urn, and it is certain that he has drawn not less than n, and certain that he has drawn not more than n; then of course it is certain that he has drawn *exactly* n. Now suppose it probable (say the odds are two to one) that he has drawn not less than n, and two to one that he has drawn not more than n; find the probability that he has drawn *exactly* n.

Solution by X. U. J.

More generally; suppose the odds are p to 1 that he has drawn not less than n, and q to 1 that he has drawn not more than n.

Let x = the probability that the number exceeds n;
y = the probability that it is equal to n;
z = the probability that it is less than n:

then $\quad x+y+z=1, \quad x+y : z = p : 1, \quad z+y : x = q : 1.$

Therefore $$x = \frac{1}{1+q}, \quad y = \frac{pq-1}{(1+p)(1+q)}, \quad z = \frac{1}{1+p}.$$

In the proposed case $p = q = 2$, and we get $x = y = z = \frac{1}{3}$.

[The PROPOSER remarks that the chances are evidently all equal that he has drawn more than n, less than n, and exactly n.]

1651. (Proposed by J. GRIFFITHS, M.A.)—Let l, m, n be the middle points of the sides BC, CA, AB of any triangle ABC; P the point of intersection of its three perpendiculars; p, q, r the middle points of the segments AP, BP, CP. Through l, m, n; p, q, r; two sets of three lines are drawn parallel to the external bisectors of the angles A, B, C, respectively, so as to form two new triangles. Prove that the sides of these triangles, together with those of the triangle ABC, are bisected by one and the same circle.

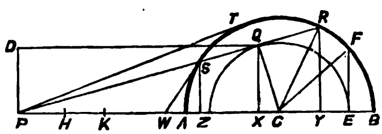
Solution by E. FITZGERALD; REV. R. TOWNSEND, M.A.; J. DALE; *and others.*

The nine-point circle of the triangle ABC circumscribes each of the triangles lmn, pqr; moreover the corresponding sides of these three triangles are obviously parallel to each other, whence it follows that the sides of the "two new triangles" are the external bisectors of the angles of the triangles lmn, pqr. Now the circle drawn round a triangle bisects the sides of the escribed triangle (that, namely, which joins the centres of the escribed circles of the primitive, or the one formed by the external bisectors of its angles); whence the truth of the theorem is evident.

1660. (Proposed by P. W. FLOOD.)—From a given point in the diameter (produced) of a given semicircle, to draw a straight line, cutting the circum-

ference in two points, such that, if perpendiculars be drawn therefrom on the diameter, (1) the rectangle contained by these perpendiculars, or (2) their sum, or (3) their difference, or (4) their ratio, may be given.

Solution by ALPHA; J. DALE; *the* PROPOSER; E. FITZGERALD; *and others.*



Let P be the given point; C the centre of the given semicircle; RY and SZ the perpendiculars in question.

(1.) When RY.SZ ($=PD^2$ suppose) is given; place PD perpendicular to PC; draw PT touching the given semicircle; and from P as centre, with PT as radius, draw a circle cutting a parallel to PC through D in Q; then PQR will be the line required. For we have $RY : PR = SZ : PS = QX : PQ = PD : PT$, $\therefore RY . SZ : PR . PS = PD^2 : PT^2$; but $PR . PS = PT^2$; $\therefore RY . SZ = PD^2$.

(2.) When RY + SZ ($=2PD$ suppose) is given; let the parallel through D meet a semicircle on PC in Q; then PQR will be the required line; for Q would then be the middle point of RS, and therefore $RY + SZ = 2QX = 2PD$.

(3.) When RY − SZ ($=PH$ suppose) is given; construct the right-angled triangle CEF, having a radius CF of the semicircle for its hypotenuse, and its area equal to $\frac{1}{4}PC . PH$: with CE as radius, draw a circle round C; then a tangent PQR from P to this circle will be the line required. For we have

$$PC : CQ = RS : RY - SZ; \quad \therefore PC\,(RY - SZ) = 2CE . EF = PC . PH;$$

therefore $\qquad RY - SZ = PH =$ the *given* difference.

It is clear that RY − SZ will be a *maximum* when the area of the triangle CEF is a maximum, that is, when CE = EF.

(4.) When RY : SZ ($=PK : PH$ suppose) is given; find a point W in PC, and from it inflect a line WS to the semicircle, so that $PC : PW = AC : WS = PK : PH$; then PSR will be the required line: for CR will be *parallel* to WS, and therefore $RY : SZ = PR : PS = PC : PW = PK : PH$.

1662. (Proposed by H. BUCKLEY.)—Let there be n circles given in position, $n-1$ straight lines may be found, such that if from any point in the circumference of one of the circles perpendiculars be drawn on the straight lines, and tangents be drawn from the same point to the circles, the product of the tangents shall always be a mean-proportional between a certain given magnitude and the product of the perpendiculars.

Solution by the REV. R. TOWNSEND, M.A.; J. DALE; E. FITZGERALD; *and others.*

Let O be the centre of the nth circle; A, B, C, &c. the centres of the remaining $n-1$; L, M, N, &c. the several radical axes of the circle (O) with

the circles (A), (B), (C), &c., respectively; P any arbitrary point on the circle (O); PL, PM, PN, &c., the perpendiculars from P on L, M, N, &c.; and PQ, PR, PS, &c., the several tangents from P to the circles (A), (B), (C), &c.; then $PQ^2 = 2OA \, . \, PL$, $PR^2 = 2OB \, . \, PM$, $PS^2 = 2OC \, . \, PN$, &c. (Townsend's *Modern Geometry*, Vol. I., Art. 182);

therefore $PQ^2 \, . \, PR^2 \, . \, PS^2 \ldots\ldots = (2OA \, . \, 2OB \, . \, 2OC \ldots)(PL \, . \, PM \, . \, PN \ldots\ldots)$.

Hence the $n-1$ radical axes L, M, N, &c. are the lines to be found.

1667. (Proposed by PROFESSOR SYLVESTER.)—
Show that the discriminant of the form

$$ax^5 + b\lambda x^4y + c\lambda^2x^3y^2 + c\mu^2x^2y^3 + b\mu xy^4 + ay^5$$

will be a rational integral function of the quantities a, b, c, $\lambda\mu$, $\lambda^5+\mu^5$, and of the second degree only in respect to the last of them.

Solution by PROFESSOR CAYLEY.

In general

$$\text{Disct.}\,(a, b, c, d, e, f)\,(\lambda x + \mu y, \lambda' x + \mu' y)^5$$
$$= (\lambda\mu' - \lambda'\mu)^{20}\,\text{Disct.}\,(a, b, c, d, e, f)\,(x, y)^5.$$

Hence first, if $(\lambda, \mu, \lambda', \mu') = (0, 1, 1, 0)$,

$$\text{Disct.}\,(a, b, c, d, e, f)\,(y, x)^5 = \text{Disct.}\,(a, b, c, d, e, f)\,(x, y)^5;$$

and secondly, if ω be an imaginary fifth root of unity and

$$(\lambda, \mu, \lambda', \mu') = (\omega, 0, 0, 1),$$

$$\text{Disct.}\,(a, b, c, d, e, f)\,(\omega x, y)^5 = \text{Disct.}\,(a, b, c, d, e, f)\,(x, y)^5.$$

These two results may also be written,

$$\text{Disct.}\,(a, b, c, d, e, f)\,(x, y)^5 = \text{Disct.}\,(f, e, d, c, b, a)\,(x, y)^5,$$

$$\text{Disct.}\,(a, b, c, d, e, f)\,(x, y)^5 = \text{Disct.}\,(a, b\omega^4, c\omega^3, d\omega^2, e\omega, f)\,(x, y)^5;$$

that is, the discriminant of $(a, b, c, d, e, f)\,(x, y)^5$ is not altered by taking the coefficients in a reverse order, or by multiplying the several coefficients by the powers ω^5, ω^4, ω^3, ω^2, ω, of an imaginary fifth root of unity. Applying these theorems to the form $(a, b\lambda, c\lambda^2, c\mu^2, b\mu, a)\,(x, y)^5$, the discriminant is not altered by changing the coefficients into $(a, b\mu, c\mu^2, c\lambda^2, b\lambda, a)$; that is, by the interchange of λ and μ; nor by changing the coefficients into

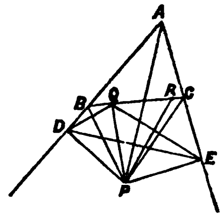
$$(a, b\omega^4\lambda, c\omega^3\lambda^2, c\omega^2\mu^2, b\omega\mu, a), \text{ or } [a, b\,(\lambda\omega^4), c\,(\lambda\omega^4)^2, c\,(\mu\omega)^2, b\,(\mu\omega), a];$$

that is, the discriminant is not altered by the change of λ, μ into $\lambda\omega^4$, $\mu\omega$ respectively. The discriminant is therefore a rational and integral function, symmetrical in regard to λ, μ, and which is not altered by the change of λ, μ into $\lambda\omega^4$, $\mu\omega$ respectively. In virtue of the second property the discriminant is a rational integral function of $(\lambda\mu, \lambda^5, \mu^5)$, and then in virtue of the first property it is a rational integral function of $(\lambda\mu, \lambda^5\mu^5, \lambda^5+\mu^5)$, that is, of $\lambda\mu$, $\lambda^5+\mu^5$. For the general form $(a, b, c, d, e, f)\,(x, y)^5$, if a term of

the discriminant be $a^\alpha b^\beta c^\gamma d^\delta e^\epsilon f^\phi$, then we have $\alpha+\beta+\gamma+\delta+\epsilon+\phi=8$, $5\alpha+4\beta+3\gamma+2\delta+\epsilon=20$; hence attending only to the indices α, β, γ we have $5\alpha+4\beta+3\gamma>20$, and therefore *à fortiori* $3\beta+3\gamma>20$, so that $\beta+\gamma$ is $=6$ at most. Hence for the form $(a, b\lambda, c\lambda^2, c\mu^2, b\mu, a)(x, y)^5$, the sum of the indices of $b\lambda$, $c\lambda^2$ is $=6$ at most, and therefore, even if the index of $c\lambda^2$ is $=6$, the index of λ will be only $=12$, that is, the discriminant contains no power of λ higher than λ^{12}; hence considered as a function of $\lambda\mu$, $\lambda^5+\mu^5$, the highest power of $\lambda^5+\mu^5$ is $(\lambda^5+\mu^5)^2$; which completes the theorem.

1670. (Proposed by PATRICK O'CAVANAGH.)—A given angle BPC turns round a fixed point P within a given angle BAC; find the locus of the foot Q of the perpendicular PQ drawn from P on the common chord BC of the two angles.

I. *Solution by* H. MURPHY; F. D. THOMSON, M.A.; J. DALE; *and others.*

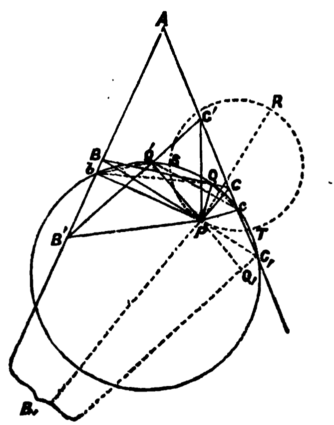


Draw PD, PE perpendicular to AB, AC; then, since a circle may be drawn round each of the quadrilaterals PQBD, PQCE, we have

$$\angle\, \text{DQE} = \text{DBP} + \text{ECP} = \text{BAC} + \text{BPC}$$
$$= \textit{a constant angle};$$

hence the locus of Q is a circular segment on DE containing a given angle.

II. *Solution by* ARCHER STANLEY.



Let BPC, B'PC' be any two positions of the moveable angle of constant magnitude; PQ, PQ' the perpendiculars let fall upon the chords BC, B'C' respectively common to these angles and the fixed angle BAC, and upon the sides of the latter let fall the perpendiculars Pb, Pc. The four points PQBb obviously lie on a circle; hence the angles BPQ and BbQ are equal to each other. For a similar reason the angle CPQ is equal to CcQ; so that the angle BPC is equal to the sum of the angles AbQ and AcQ. But by hypothesis it is also equal to B'PC', which latter may in like manner be proved to be equal to the sum of AbQ' and AcQ'. Deducting from each of these equal sums the angles AbQ' and AcQ common to both, the remainders QbQ', QcQ' will be equal, and consequently Q' will lie on the circle which

passes through, and is determined by, b, c, and Q. This circle, therefore, is the locus required. It will break up into bc and the line at infinity if the constant angle BPC should be equal to the supplement of A.

It will be observed that the circle bQc is the first positive pedal of the envelope of the chord BC common to the fixed and moveable angles. This envelope is a conic Σ to which AB and AC are tangents; for PB, PC being, obviously, corresponding rays of *equal* pencils, BC is the connector of corresponding points of homographic ranges on AB, AC. The point P must be the focus of the conic Σ, otherwise the pedal could not be a circle. In fact, the tangents from P to Σ are manifestly the double rays of the equal pencils described by PB, PC, and the latter are well known to be the imaginary lines which connect P with the circular points at infinity. (*Géométrie Supérieure*, Art. 651.)

It may be readily shown that, conversely: *If perpendiculars* bA, cA *be erected at the extremities of two lines* Pb, Pc *drawn from a given point* P *to a circle* bcQ, *they will intercept on the perpendiculars at the extremities of all other lines* PQ, PQ′, *&c., from* P *to the circle, segments* BC, B′C′ *which will subtend at* P *a constant angle.*

In fact, if to the equal angles QbQ', QcQ' we add the sum of the angles AbQ' and AcQ we obtain the equal sums $AbQ + AcQ$ and $AbQ' + AcQ'$, which, as before shown, are respectively equal to BPC, B′PC′.

Hence may be deduced the well known theorem, that *Two fixed tangents of a conic intercept on any other tangent a segment which subtends a constant angle at the focus.*

1671. (Proposed by MATTHEW COLLINS, B.A.)—Through a given point R to draw a straight line BRC, meeting the sides of a given angle BAC in B and C, so that BC may subtend a given angle at another given point P.

I. *Solution by* ALPHA; H. MURPHY; J. DALE; F. D. THOMSON, M.A.; *and others.*

Draw PD, PE, PQ perpendicular respectively to BA, AC, CB (*see Fig. to Quest.* 1670); then it is shown in the foregoing solution that the locus of Q is a given circular segment on the chord DE; hence the positions of BRC will be determined by the intersections of this segment with a *circle on* PR *as diameter.*

II. *Solution by* ARCHER STANLEY.

Let BPC be equal to the given angle, and PQ, Pb, Pc be respectively perpendicular to BC, AB, AC. On PR as diameter describe a circle cutting in S (and T) the circle drawn through Q, b, and c. Then RS (or RT) will be the line required. For from my solution to Quest. 1670, it follows that AB, AC, perpendicular to Pb, Pc, intercept on every line SR (drawn through a point S, on the circle bQc, perpendicular to PS) a segment which subtends at P an angle equal the given angle BPC.

Other two solutions will be obtained by producing BP, CP to C_1, B_1 respectively, letting fall the perpendicular PQ_1 on B_1C_1 and drawing the circle Q_1bc to cut that on PR in S_1, T_1.

1679. (Proposed by H. R. GREER, B.A.)—To find the envelope of the straight line joining the feet of the perpendiculars drawn on the sides of a triangle from a point in the circumference of the circumscribed circle.

Solution by W. K. CLIFFORD.

It shall be shown that the envelope here required is identical with that required in the first part of Question 1680.

The line in question is known to be the tangent at the vertex of a parabola which touches the sides of the triangle. Now this straight line, being always at right angles to the axis of the parabola, determines on the line at infinity a series of points in involution with the series determined by the parabola itself; we have then a series of conics touching four given lines, and a series of points on one of the lines, homographic with the series of conics; and we want to find the envelope of the remaining tangent, drawn from each point to its corresponding conic. Let then $U = kV$ be the tangential equation of the series of conics, and $P = kQ$ of the series of points. We obtain the required envelope by eliminating k; it is $UQ = VP$, a curve of the third class touching the common tangents of U and V, and the line PQ. When, as in the case we are considering, the line PQ coincides with one of the common tangents of U, V, then it is a double tangent to the curve $UQ = VP$, and the points of contact are the double points of the involution; in this case, the circular points at infinity. Since the curve is of the third class, and has one double tangent (that is, all it can) it is of the fourth order; and because the double tangent has imaginary contacts, the curve has three *real* cusps. To determine the position of these cusps, and the general form of the curve, we have to study a most singular figure.

Consider four points, 1, 2, 3, 4, such that each is the intersection of perpendiculars of the triangle formed by the other three. About the triangles 234, 341, 412, 123 describe circles; it is known that these circles are all equal, and that their centres 1′2′3′4′ form another quadrangle, exactly similar and equal to 1234, but in an inverted position, their centre of (inverse) similitude being the centre of the nine-point circle. Now suppose that the feet of the perpendiculars from any point in the circle 234 to the sides of the triangle 234 are joined by a line X. Then I say that *if at the points where the line* X *cuts the six connectors of the quadrangle* 1234, *perpendiculars be drawn to these six connectors respectively; the perpendiculars will concur three by three, in four points* 1″, 2″, 3″, 4″, *situate one on each of the four circumscribing circles, and forming a quadrangle equal, similar, and similarly situated to* 1′2′3′4′. And *the centre of* (inverse) *similitude of* 1234 *and* 1″2″3″4″ *is situated on the line* X, *and bisects the segment determined on it by any pair of connectors.* Hence we see (1), that the line X is connected with the whole quadrangle, and not with three particular points of it; (2), it is cut by the connectors in an involution, one double point of which is at infinity; and therefore is an *asymptote* of some conic passing through the points 1234.

Now, take any connector 12, and find a point on it, symmetrical in respect of 1, 2, with the point where it is cut by 34. Then the envelope of X touches all the connectors at the points thus determined.

Since writing the above, I have read a paper on the subject, by Steiner, in the 53rd volume of Crelle's *Journal*. He asserts that the curve is a hypocycloid of 3 branches, and gives a simple construction for the cusps.

The property of a quadrangle enunciated above, is in fact this;—If four parabolas be drawn, having their axes parallel, each inscribed in one of the

four triangles determined by a quadrangle, these four will have a common tangent; which is at once seen to be a particular case of the reciprocal of this:—The four circles, each circumscribing one of the triangles determined by a quadrilateral, have a common point. And this again is a particular case of that wonderful proposition, the involution of cubics:—All the cubics which pass through eight fixed points pass also through a ninth point.

Finally, reciprocate the whole figure in respect of the self-conjugate circle of any of the triangles 234, &c. We thus get the locus of a point where the normal at (1) meets again a rectangular hyperbola circumscribing the quadrangle; it is a cubic having its asymptotes parallel to the sides of 234, and with a double point at (1), the tangents to which are the asymptotes of the polar circle. In fact, this problem is rather easier than its reciprocal.

1680. (Proposed by F. D. Thomson, M.A.)—

(1) Prove that the envelope of the asymptotes of a rectangular hyperbola described about a given triangle is a curve of the third class, touching the sides of the triangle, the three perpendiculars, lines through the feet of the perpendiculars parallel to the opposite sides of the triangle formed by joining them, and also the line at infinity.

(2) Prove that the envelope of the asymptotes of a conic inscribed in a given quadrilateral, is a curve of the third class touching the sides and diagonals of the quadrilateral, the line at infinity, and the line joining the middle points of the diagonals.

I. *Solution by* W. K. Clifford.

(1.) It is shown in the Solution of the preceding Question (1679) that the line whose envelope is there considered is an asymptote of *some* rectangular hyperbola circumscribing the quadrangle; whence the two envelopes must be identical. This may also be proved thus: the proposition is that a rectangular hyperbola may circumscribe any triangle which circumscribes a parabola, and have for an asymptote the tangent at the vertex of the parabola. Let β be the axis of the parabola, α the tangent at its vertex, γ the line at infinity; then the respective equations to the hyperbola and parabola are

$$\gamma^2 + 2p\alpha\gamma = 2\mu\alpha\beta, \quad \beta^2 = 2\lambda\gamma\alpha;$$

whence $\Theta = -p^2$, $\Theta' = 2p\lambda$, $\Delta = -\mu^2$, $\Delta' = -\lambda^2$, and the condition $\Theta'^2 = 4\Theta\Delta'$ is satisfied. In fact, the triangle $(\alpha\gamma\gamma)$ is inscribed in the hyperbola and circumscribes the parabola.

Hence (i.) the envelope of the asymptotes of all conics through four given points is a three-cusped quartic touching the six connectors of the given points, and the line at infinity at the points of contact of parabolæ through them. (ii.) If two tangents to a three-cusped quartic divide harmonically the double tangent, their intersection lies on a conic through the points of contact of the double tangent. This conic touches the quartic in three points. (iii.) If a chord of a nodal cubic subtend harmonically the double point, its envelope is a conic touching the tangents at the double point, and the curve itself in three points.

M. Chasles gets the result (i.) by his method of characteristics. (Theor.

XVI.) The envelope of the asymptotes is in general of class $\mu+\nu$, and has a ν-ple tangent at infinity; where μ is the number of conics of a system that can be drawn through a given point, and ν the number that can be drawn to touch a given line.

(2.) Here again M. Chasles' method shows that the envelope is of the third class, and touches *once* the line at infinity. Let U, V be two inscribed conics, and (ξ, η, ζ) the coordinates of the line at infinity; and write also Δ for $(\xi\delta_x+\eta\delta_y+\zeta\delta_z)$; then a conic of the system is $U=kV$, the centre $\Delta U=k\Delta V$, and the envelope required $U\Delta V=V\Delta U$, which is of the third class, touching the sides of the quadrilateral, and the line $\Delta U=0$, $\Delta V=0$, which joins the middle points of the diagonals. If for U, V we write AB, CD, the equation is $AB(C.\Delta D+D.\Delta C)=CD(A.\Delta B+B.\Delta A)$, showing that the curve touches the lines $(A=0, B=0)$ and $(C=0, D-0)$; that is, the diagonals of the quadrilateral.

II. *Solution by the* PROPOSER.

(1.) To find the envelope of the asymptotes of a rectangular hyperbola described about a given triangle.

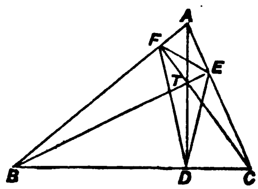


Fig. 1.

Let ABC (Fig. 1) be the given triangle, T the intersection of the perpendiculars. Take D, E, F the feet of the perpendiculars as the points of reference. Then since any rectangular hyperbola about ABC passes through T, it is easily seen that the triangle DEF is self-conjugate with reference to the conic, and therefore its equation in *tangential* coordinates is of the form

$$lx^2+my^2+nz^2=0 \quad \text{(i.)}$$

Now if x', y', z' be the coordinates of a tangent to (i.), the equation to the point of contact is given by

$$lxx'+myy'+nzz'=0 \quad \text{(ii.)}$$

But since T is on the curve, the equation to T may be made identical with (ii.); and the equation to T is

$$a'x+b'y+c'z=0 \quad \text{(iii.)},$$

where a', b', c' are the sides EF, FD, DE, since TD, TE, TF bisect the angles of the triangle of reference; hence, comparing (ii.) and (iii.), we have

$$\frac{lx'}{a'}=\frac{my'}{b'}=\frac{nz'}{c'}=\left(\frac{lx'^2+my'^2+nz'^2}{l^{-1}a'^2+m^{-1}b'^2+n^{-1}c'^2}\right)^{\frac{1}{2}}, \therefore \frac{a'^2}{l}+\frac{b'^2}{m}+\frac{c'^2}{n}=0 \quad \text{(iv.)}$$

the condition that (i.) may be a rectangular hyperbola. Now since an asymptote is a tangent at an infinite distance, their coordinates are found by combining the equation $\phi(x, y, z)=0$ to a curve with the equation $\frac{d\phi}{dx}+\frac{d\phi}{dy}+\frac{d\phi}{dz}=0$; therefore the asymptotes of (i.) are given by the equation (i.) and

$$lx+my+nz=0 \quad \text{(v.)}$$

Hence the *envelope* of the asymptotes will be found by eliminating l, m, n between the equations (i.), (iv.), (v.) The result, which is the equation of the required envelope, is readily found to be

$$a'^2x\,(z-x)\,(x-y)+b'^2y\,(x-y)\,(y-z)+c'^2z\,(y-z)\,(z-x)=0\;\ldots\ldots\;\text{(vi.)}$$

The equation (vi.) is evidently satisfied by $(x=0,\ y-z=0)$ a line through D parallel to FE, or by $(x=0,\ by-cz=0)$ the line BC, or by $(x=0,\ by+cz=0)$ the line DT; hence the envelope touches AB, BC, CA; the three perpendiculars; and lines parallel to the sides of the triangle of reference through D, E, F. The line at infinity is evidently another tangent.

In precisely the same manner we may find generally the envelope of the asymptotes of a conic described about a given quadrilateral.

(2.) To find the envelope of the asymptotes of a conic inscribed in a given quadrilateral.

The general equation to the quadrilateral is of the form $xz=kyt$, and the asymptotes are given by combining this equation with $x+z=k\,(y+t)$; hence the envelope of the asymptotes is

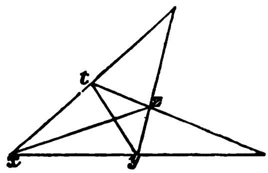


Fig. 2.

$$yt\,(x+z)=xz\,(y+t),\quad\text{or}\quad xy\,(t-z)=zt\,(x-y).$$

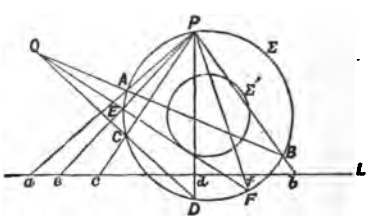
We see, therefore, that the envelope is a curve of the third class, touching the four sides of the quadrilateral, the two diagonals, the line joining the middle points of the diagonals, and the line at infinity.

This envelope may be compared with the locus in Question 1562. (*Reprint*, Vol. II., p. 70.)

1686. (Proposed by the EDITOR.)—Through four given points to draw a conic such that the chord which it intercepts on a fixed line shall (1) have a given length, or (2) shall subtend a given angle at a given point.

I. *Solution by* ARCHER STANLEY.

The several conics which pass through the four given points cut the given line L in pairs of points in involution. Three of these conics consist of pairs of right lines, and the involution is determined by the intersections with L of any two of these three pairs of lines. The problem, therefore, is reduced to this:—*Given two pairs of conjugate points a, b and c, d of an involution, to find a third pair e, f, such that the segment ef, shall* (1) *have a given length, or* (2) *shall subtend a given*



angle at a given point P. The conic passing through e, f and the four given points will be the one required.

Of each of the problems (1), (2), there are two solutions. A simple construction for (1) is the following:—On ab and cd as diameters describe two circles. If the latter cut one another, then with either intersection as centre and half the given length as radius describe a circle cutting L in m and m'. A circle equal to the last, and having either m or m' for centre, will be coaxal with the circles on ab, cd; and will, consequently, cut L in a pair of points e, f, forming, with a, b and c, d an involution, and intercepting on L a segment ef of the required length. If the circles on ab and cd should not cut one another, it will still be easy to find the two coaxal circles whose diameters have the required magnitude.

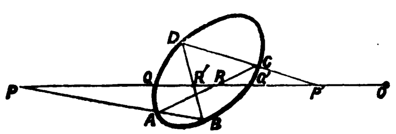
The problem (2) may be thus solved:—Draw any circle (Σ) whatever through the point P, and let A, B, C, D be the points in which it is cut by the connectors Pa, Pb, Pc, Pd. Join AB and CD to intersect in Q.

Concentric with Σ draw a circle Σ' touching any chord of Σ which subtends at P an angle of the given magnitude. From Q draw a tangent to Σ' cutting Σ in E and F, and project the latter points, from P, on the given line L. The projections e and f will be the points required. For ef manifestly subtends the required angle at P; and e, f, by a well known theorem, are conjugate points of the involution determined by the pairs a, b and c, d.

The line L being at infinity, this problem enables us to draw through four given points the two hyperbolas whose asymptotes enclose a given angle.

II. *Solution by* F. D. THOMSON, M.A.

(1.) To draw a conic through four given points such that the chord which it intercepts on a fixed line may have a given length.



Let the fixed line meet the conic in Q, Q′, and the sides and diagonals of the quadrilateral in P, P′; R, R′. Then P, P′; Q, Q′; R, R′ form a series of points in involution. Find O the centre of involution (Salmon's *Conics*, Art. 338), by means of the known points P, P′, R, R′;

then $\quad OQ \,.\, OQ' = OP \,.\, OP' =$ a known magnitude,

while $\quad QQ' = OQ - OQ' =$ a given length.

Hence we get a quadratic to determine OQ or OQ′, and the conic through ABCDQ is the one required.

(2.) When the intercept subtends a given angle at a given point. Let X be the given point; then

$$\{PQRP'\} = \{P'Q'R'P\}, \;\therefore\; \frac{\sin PXR}{\sin QXR} : \frac{\sin PXP'}{\sin QXP'} = \frac{\sin P'XR'}{\sin Q'XR'} : \frac{\sin P'XP}{\sin Q'XP},$$

$$\therefore\; \sin PXR \,.\, \sin QXP' \sin Q'XR' = \sin P'XR' \,.\, \sin Q'XP \sin QXR.$$

Now PXR, P′XR′, QXQ′ are known angles, and the remaining angles can be expressed in terms of PXQ and known angles; this gives an equation which determines PXQ, and therefore determines the conic.

[Mr. TOWNSEND's solution is, in short, as follows :—

A variable conic passing through the four fixed points determines two homographic divisions in involution on the fixed line; and the problem is consequently (by a well known property of involution) reduced to the corresponding but simpler particular case of itself, viz., to draw a circle passing through two given points and intercepting on a given line a segment (1) of given length, or (2) subtending at either point an angle of given magnitude, the solutions of which are evident.]

1620. (Proposed by the Rev. J. BLISSARD.)—Let fx be any function of x capable of expansion in terms of x, and let $f_n x$ denote $\left(\frac{d}{dx}\right)^n fx$; and therefore $f_n 0 = \left(\frac{d}{dx}\right)^n fx$ (when $x=0$); then it is required to prove that

$$fx = f0 + \frac{m}{m+n} \cdot f_1 0 \cdot \frac{x}{1} + \frac{m(m+1)}{(m+n)(m+n+1)} \cdot f_2 0 \cdot \frac{x^2}{1.2} + \&c.$$
$$+ \frac{n}{m+n} \cdot f_1 x \cdot \frac{x}{1} - \frac{n(n+1)}{(m+n)(m+n+1)} \cdot f_2 x \cdot \frac{x^2}{1.2} + \&c.$$

where m and n are perfectly arbitrary.

Solution by the PROPOSER.

Let $U_n = n(n-1)\ldots\ldots(n-r+1)$, r being a positive integer; then, using *Representative Notation*,

$$U^m (U-1)^n fUx = U^m (U-1)^n f\{x + (U-1)x\} \ldots\ldots\ldots\ldots (A).$$

Hence, expanding the left-hand side of (A) by Maclaurin's Theorem, and the right-hand side by Taylor's Theorem, we have

$$U^m (U-1)^n \left\{f0 + f_1 0 \cdot \frac{Ux}{1} + f_2 0 \cdot \frac{U^2 x^2}{1.2} + \ldots\ldots\right\}$$
$$= U^m (U-1)^n \left\{fx + f_1 x \cdot \frac{(U-1)x}{1} + f_2 x \cdot \frac{(U-1)^2 U^2}{1.2} + \ldots\ldots\right\}.$$

But it has been shown (see Solution of Quest. 1567, *Reprint*, Vol. III., p. 11) that $U^m (U-1)^n = \dfrac{\Gamma(r+1)\,\Gamma(m+1)}{\Gamma(r+1-n)\,\Gamma(m+n+1-r)}$; hence, making this substitution, we have

$$\frac{\Gamma(r+1)}{\Gamma(r+1-n)}\left\{\frac{\Gamma(m+1)}{\Gamma(m+n+1-r)}\cdot f0+\frac{\Gamma(m+2)}{\Gamma(m+n+2-r)}\cdot f_1 0\cdot\frac{x}{1}\right.$$
$$\left.+\frac{\Gamma(m+3)}{\Gamma(m+n+3-r)}\cdot f_2 0\cdot\frac{x^2}{1.2}+\&c.\right\}$$
$$=\Gamma(r+1)\,\Gamma(m+1)\left\{\frac{fx}{\Gamma(r+1-n)\,\Gamma(m+n+1-r)}+\right.$$
$$\left.\frac{f_1x}{1}\cdot\frac{x}{\Gamma(r-n)\,\Gamma(m+n+2-r)}+\frac{f_2x}{1.2}\cdot\frac{x^2}{\Gamma(r-n-1)\,\Gamma(m+n+3-r)}+\&c.\right\}$$

$$\therefore fx=f0+\frac{m+1}{m+n+1-r}\cdot f_1 0\cdot\frac{x}{1}+\frac{(m+1)(m+2)}{(m+n+1-r)(m+n+2-r)}\cdot f_2 0\cdot\frac{x^2}{1.2}+\dots$$
$$-\left\{\frac{r-n}{m+n+1-r}\cdot f_1x\cdot\frac{x}{1}+\frac{(r-n)(r-n-1)}{(m+n+1-r)(m+n+2-r)}\cdot f_2x\cdot\frac{x^2}{1.2}+\dots\right\}.$$

For n put $n+r$, and for m put $m-1$; then we obtain

$$fx=f0+\frac{m}{m+n}\cdot f_1 0\cdot\frac{x}{1}+\frac{m(m+1)}{(m+n)(m+n+1)}\cdot f_2 0\cdot\frac{x^2}{1.2}+\dots$$
$$+\frac{n}{m+n}\cdot f_1x\cdot\frac{x}{1}-\frac{n(n+1)}{(m+n)(m+n+1)}\cdot f_2x\cdot\frac{x^2}{1.2}+\dots$$

COROLLARY.—Let $fx=\log(1+x)$; then, by the above theorem, we have

$$\log(1+x)=\frac{m}{m+n}\cdot\frac{x}{1}-\frac{m(m+1)}{(m+n)(m+n+1)}\cdot\frac{x^2}{2}+\&c.$$
$$+\frac{n}{m+n}\left(\frac{x}{1+x}\right)+\frac{1}{2}\,\frac{n(n+1)}{(m+n)(m+n+1)}\left(\frac{x}{1+x}\right)^2+\&c.$$

Now put $x=i$, where $i=\sqrt{(-1)}$ as usual; then, since

$$\log(1+i)=i\,(1-\tfrac{1}{3}+\tfrac{1}{5}-\&c.)+\tfrac{1}{2}(1-\tfrac{1}{2}+\tfrac{1}{3}-\&c.)=\tfrac{1}{4}\pi i+\tfrac{1}{2}\log 2,$$

by equating real and unreal quantities we have two equations, one giving a generalized form for log 2, the other a generalized form for $\frac{1}{4}\pi$. The latter is

$$\frac{\pi}{4}=\frac{m}{m+n}-\frac{1}{3}\cdot\frac{m(m+1)(m+2)}{(m+n)(m+n+1)(m+n+2)}+\frac{1}{5}\cdot\frac{m\dots(m+4)}{(m+n)\dots(m+n+4)}-\&c.$$
$$+\left\{\frac{n}{m+n}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{n(n+1)}{(m+n)(m+n+1)}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{n\dots(n+2)}{(m+n)\dots(m+n+2)}\cdot\frac{1}{2^2}\right\}$$
$$-\left\{\frac{1}{5}\cdot\frac{n\dots(n+4)}{(m+n)\dots(m+n+4)}\cdot\frac{1}{2^3}+\frac{1}{6}\cdot\frac{n\dots(n+5)}{(m+n)\dots(m+n+5)}\cdot\frac{1}{2^3}\right.$$
$$\left.+\frac{1}{7}\cdot\frac{n\dots(n+6)}{(m+n)\dots(m+n+6)}\cdot\frac{1}{2^4}\right\}+\&c.$$

[NOTE.—A *verification* of the theorem in the Question may be obtained as follows :—

Expanding fx, f_1x, f_2x, &c. by Maclaurin's theorem, we have

$$fx - \frac{n}{m+n} \cdot f_1x \cdot \frac{x}{1} + \frac{n(n+1)}{(m+n)(m+n+1)} \cdot f_2x \cdot \frac{x^2}{1 \cdot 2} - \&c. =$$

$$f0 + C_1 \cdot f_1 0 \cdot \frac{x}{1} + C_2 \cdot f_2 0 \cdot \frac{x^2}{1 \cdot 2} + \ldots + C_r \cdot f_r 0 \cdot \frac{x^r}{1 \cdot 2 \ldots r} + \&c. \ldots \text{(A)},$$

where $\quad C_r = 1 - \frac{r}{1} \cdot \frac{n}{m+n} + \frac{r(r-1)}{1 \cdot 2} \cdot \frac{n(n+1)}{(m+n)(m+n+1)} - \&c.$

Now it can be easily proved that

$$1 - \frac{r}{1} \cdot \frac{n}{m+n} + \frac{r(r-1)}{1 \cdot 2} \cdot \frac{n(n+1)}{(m+n)(m+n+1)} - \&c. \ (= C_r)$$

$$= \frac{m(m+1) \ldots (m+r-1)}{(m+n)(m+n+1) \ldots (m+n+r-1)} \ldots \ldots \ldots \text{(B)};$$

for suppose (B) to be true for any assigned value of r, then, taking the consecutive value, we have

$$C_{r+1} = 1 - \frac{r+1}{1} \cdot \frac{n}{m+n} + \frac{(r+1)r}{1 \cdot 2} \cdot \frac{n(n+1)}{(m+n)(m+n+1)} - \&c.$$

$$= \left\{ 1 - \frac{r}{1} \cdot \frac{n}{m+n} + \frac{r(r-1)}{1 \cdot 2} \cdot \frac{n(n+1)}{(m+n)(m+n+1)} - \&c. \right\}$$

$$- \frac{n}{m+n} \left\{ 1 - \frac{r}{1} \cdot \frac{n+1}{m+n+1} + \frac{r(r-1)}{1 \cdot 2} \cdot \frac{(n+1)(n+2)}{(m+n)(m+n+2)} - \&c. \right\}$$

$$= \frac{m(m+1) \ldots (m+r-1)}{(m+n)(m+n+1) \ldots (m+n+r-1)}$$

$$- \frac{n}{m+n} \cdot \frac{m(m+1) \ldots (m+r-1)}{(m+n+1)(m+n+2) \ldots (m+n+r)}$$

$$= \frac{m(m+1) \ldots (m+r)}{(m+n)(m+n+1) \ldots (m+n+r)}.$$

If, therefore, (B) is true for any value of r, it is also true for the consecutive value; but it is seen to be true when $r=1$, therefore it is likewise true for $r=2, 3, 4$, &c., that is to say, for *all* such (integral) values of r.

Thus (A) becomes

$$fx - \frac{n}{m+n} \cdot f_1x \cdot \frac{x}{1} + \frac{n(n+1)}{(m+n)(m+n+1)} \cdot f_2x \cdot \frac{x^2}{1 \cdot 2} - \&c. =$$

$$f0 + \frac{m}{m+n} \cdot f_1 0 \cdot \frac{x}{1} + \frac{m(m+1)}{(m+n)(m+n+1)} \cdot f_2 0 \cdot \frac{x^2}{1 \cdot 2} + \&c.,$$

which (after transposition) is the theorem in the Question.

On sending the foregoing investigation to Mr. Blissard, he stated that he had given a "proof of the property of Numbers marked (B)—a very important property which enters largely into many analytical generalisations—in No. 22, pp. 168, 169, of the *Quarterly Journal of Mathematics*."

Mr. Blissard adds that his "proof of this theorem, obtained by the aid of *Representative Notation*, determines the limits within which it holds good; the sole restriction being that, if r is not a positive integer, $(m+r)$ must be a positive quantity."—EDITOR.]

1621. (Proposed by M. W. CROFTON, B.A.)

(1.) An endless string is passed round *any* curve, and a second curve is described by a pencil which moves so as constantly to stretch the string (as one confocal ellipse may be generated from another). Let V be any position of the pencil; VA, VB the portions of string which are tangents to the inner curve; an ellipse through V, with A, B as foci, has contact of the *second order* with the locus of V.

(2.) From any point V on an ellipse tangents VA, VB are drawn to any confocal ellipse. If now from A, B as foci an ellipse be drawn through V, it will have contact of the *third order* with the first ellipse.

Solution by the PROPOSER; *and* E. FITZGERALD.

1. Let WZ (Fig. 1) be a curve generated as in (1) from the curve AB; then the tangents VA, VB from any point V are equally inclined to the curve WZ. Let V′ be a consecutive point; let the new tangents V′A′, V′B′ make angles $d\theta$, $d\theta'$ with the former, VA, VB; put T, T′ for the tangents VA, VB; then

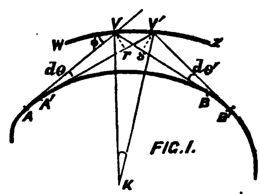


FIG. 1.

$$Vr = VV' \sin\phi = T\,d\theta,\quad V's = VV' \sin\phi = T'd\theta';$$

$$\therefore\ d\theta + d\theta' = VV' \sin\phi \left(\frac{1}{T} + \frac{1}{T'}\right).$$

But, drawing the normals VK, V′K, which bisect the angles AVB, A′V′B′, it is easy to show that $d\theta + d\theta' = 2K$; hence, if R be the radius of curvature at V, we have

$$\frac{2K}{VV'} = \frac{2}{R} = \sin\phi\left(\frac{1}{T} + \frac{1}{T'}\right),$$

which is the same expression as for an ellipse whose foci are A, B, and which passes through V: the ellipse has, therefore, then 3-pointic contact with WZ.

2. If the curves WZ, AB (Fig. 2) be two confocal ellipses, the osculating ellipse has contact of the *third* order: for, its centre M being the middle point of AB, its diameter at V coincides with that of the ellipse WZ; now if two cónics have contact of the second order, their centres being in a straight line with the point of contact, they must have a fourth consecutive point in common;* so that the contact is of the third order.

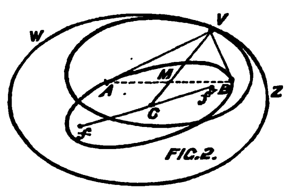


FIG. 2.

Hence it appears that the foci of the system of conics which have 4-pointic contact with an ellipse at a given point, are the points of contact of pairs of tangents from that point to conics confocal with the ellipse.

[* For if two conics touch, and have a common diameter at the point of contact, we may write their equations $x^2 = Ay + By^2$, $x^2 = Cy + Dy^2$; subtracting, we have $0 = y\left\{(A-C) + (B-D)y\right\}$; whence we see that two *intersections* are at the ends of a common chord parallel to the common tangent; so that, if they have 3-pointic contact, this chord must coincide with the tangent; and the contact will, in this case, be 4-pointic.]

1687. (Proposed by Professor CAYLEY.)—To describe a spherical triangle such that the angles thereof and of the polar triangle lie on a spherical conic.

On the sphere, the locus of a point such that the perpendiculars from it upon the sides of a given spherical triangle have their feet on a line (great circle), is in general a spherical cubic; if however the triangle be such as is mentioned in the above Problem, then the locus breaks up into a line (great circle) and into the conic through the angles of the given and polar triangles.

Solution by T. COTTERILL, M.A.

In a plane, if the angular points of two homologous triangles are on a conic, it is well known that if a line be drawn through the centre of homology, the lines joining its intersections with the sides of one triangle to the corresponding angles of the other are concurrent on the conic. And, reciprocally, the lines joining a point in the axis of homology to the angles of one triangle cut the corresponding sides of the other in three points on a tangent to the conic inscribed in the two triangles.

As this holds good on the sphere, and all great circles through a pole are at right angles to its polar, we have only to show that a triangle can be found coconical with its polar, and the problem is solved. I do not see how this is to be determined geometrically, but by the following method we shall find analytically the required condition, and also the equations to the locus and envelope.

Employing the usual notation for a triangle and its polar, and the system of coordinates explained by Dr. Salmon in the chapter of his *Solid Geometry* which throws so much light on spherical conics; we shall have, if the sine of the arcual distance between a point P and a great circle t be denoted by P_t, for the coordinates of the angles of the polar triangle $A'_a = 1$, $A'_b = -\cos C$, $A'_c = -\cos B$; and if H is the centre of homology of the triangles,

$$\cos a \,.\, \cos AH = \cos b \,.\, \cos BH = \cos c \,.\, \cos CH,$$

$$\cos A \,.\, H_a = \cos B \,.\, H_b = \cos C \,.\, H_c.$$

Hence $\Sigma(\tan a \,.\, P_a) = 0$ is the equation to the polar great circle of the point H.

The equation to the polar of A, or the side a', is

$$\sin a \,.\, P_a + \cos c \sin b \,.\, P_b + \cos b \,\sin c \,.\, P_c = 0;$$

and, therefore, the equation to the axis of homology is $\Sigma(\tan a \,.\, P_a) = 0$, or the polar of H.

The equation to the conic ABC B′C′ is

$$\frac{\tan A}{P_a} = \frac{\cos c}{\cos C} \cdot \frac{\tan B}{P_b} + \frac{\cos b}{\cos B} \cdot \frac{\tan C}{P_c},$$

and assumes the symmetrical form $\Sigma(\tan a \,.\, P_b \,.\, P_c) = 0$, if $A = \pi - a$, $B = b$, $C = c$. This, then, is the condition required in the enunciation of this interesting Problem; but in order to show that the triangle in which an angle and its opposite side are supplemental, is the only one which satisfies it, we must form the condition that the two triangles shall be on a conic, and then examine the cases in which it vanishes.

This will be found to be the remarkable determinant,

$$\begin{vmatrix} -\cos B\ \cos C, & \cos B, & \cos C \\ \cos A, & -\cos C\ \cos A, & \cos C \\ \cos A, & \cos B, & -\cos A\ \cos B \end{vmatrix} = \Sigma(\cos^2 B \cos^2 C) + 2\cos A \cos B \cos C - \cos^2 A \cos^2 B \cos^2 C$$

$$= \sin^2 B \sin^2 C (\sin^2 A - \sin^2 a) = \&c.\ ;$$

and its quasi-reciprocal and polar,

$$-\begin{vmatrix} \cos b \cos c, & \cos b, & \cos c \\ \cos a, & \cos c \cos a, & \cos c \\ \cos a, & \cos b, & \cos a \cos b \end{vmatrix} = \Sigma \cos^2 b \cos^2 c - 2\cos a \cos b \cos c - \cos^2 a \cos^2 b \cos^2 c$$

$$= \sin^2 b \sin^2 c (\sin^2 a - \sin^2 A) = \&c.$$

This shows that the condition given above (which implies that each remaining side must be equal to its opposite angle) is necessary and sufficient.

If Δ denote the last determinant, we have

$$\Delta \sec^2 a \sec^2 b \sec^2 c = \Sigma \sec^2 a - 2 \sec a \sec b \sec c - 1$$
$$= \Sigma \tan^2 a - 2(\sec a \sec b \sec c - 1).$$

By means of this expression I find, in reference to the remark at the end of the Question, that the general locus can be brought under the form

$$\Delta \,.\, \sec^2 a \,.\, P_a P_b P_c = \Sigma \tan a \,.\, P_a\, \Sigma \tan a \,.\, P_b P_c.$$

The transformation of this into the shape

$$(P_a + \cos C \,.\, P_b)(P_b + \cos A \,.\, P_c)(P_c + \cos B \,.\, P_a)$$
$$+ (P_b + \cos C \,.\, P_a)(P_c + \cos A \,.\, P_b)(P_a + \cos B \,.\, P_c) = 0$$

will illustrate the use of the above determinants. The envelope of the foot line is formed by substituting A'_t, B'_t, C'_t for P_a, P_b, P_c, respectively. The two curves are therefore supplementary.

1689. (Proposed by T. A. Hirst, F.R.S.)—Let p, p' be two variable points collinear with a fixed point A, and so situated that the segment pp' always subtends a right angle at another fixed point M.

Prove the following properties of *corresponding* loci of p and p':—

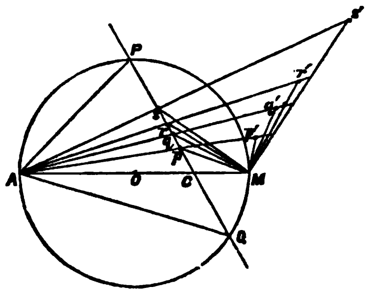
(1.) Right lines equidistant from the middle point of AM correspond to similar conics, passing through A and cutting AM perpendicularly at M.

(2.) These conics are similar ellipses, parabolas, or hyperbolas, according as the common distance of the primitive lines from the middle point of AM is greater than, equal to, or less than $\frac{1}{2}$AM.

(3.) The circles which pass through A and M, taken in pairs, constitute corresponding loci: as also do the circles which pass through M and have their centres on AM.

N.B.—On the above may be founded a method of transformation, analogous to inversion. Both methods, in fact, as I have shown in the proceedings of the Royal Society for March, are special cases of what may be termed *Quadric Inversion.*

Solution by F. D. THOMSON; *and* J. DALE.

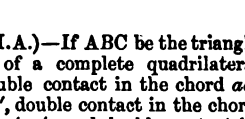


(1.) Let PQ be a straight line; p, q, r, s points on it, to which correspond the points p', q', r', s'. Then
$(M . p'q'r's') = (M . pqrs)$
$= (A . pqrs) = (A . p'q'r's')$,
therefore, $p'q'r's'$ lie on a conic through A and M. Again, considering the point adjacent to C where PQ cuts AM, we see that the tangent at M is perpendicular to AM.

Let the circle on AM as diameter cut PQ in P and Q. Then the points corresponding to P and Q will be at infinity, and therefore the asymptotes are parallel to AP and AQ. The angle between the asymptotes is therefore the angle PAQ. Hence, all straight lines which cut off similar segments PAQ have the same angle between their asymptotes; that is, they are *similar*. And straight lines which cut off similar segments are equidistant from O the centre of the circle.

(2.) The asymptotes will be real or imaginary, that is, the conic will be an ellipse or hyperbola, according as PQ cuts the circle in real or imaginary points; and the conic will be a parabola if PQ *touches* the circle, for then there is only one direction in which there is a point at an infinite distance on the curve.

(3.) It is evident that if the angle ApM is constant, $Ap'M$ is also constant. If, therefore, the locus of p is a circle through A and M, the locus of p' will also be a circle through A and M.



If p be on a circle whose centre is C and diameter BM, Mp' is parallel to Bp and bears a constant ratio to it; hence the locus of p' is similar to that of p, and is therefore a circle having its centre on BM produced.

1690. (Proposed by W. A. WHITWORTH, M.A.)—If ABC be the triangle formed by the three diagonals aa', bb', cc' of a complete quadrilateral $aa'bb'cc'$, then a conic can be found having double contact in the chord aa' with the critical conic of the quadrilateral $bb'cc'$, double contact in the chord bb' with the critical conic of the quadrilateral $cc'aa'$, and double contact in the chord cc' with the critical conic of the quadrilateral $aa'bb'$.

The same conic will also intersect in the chord $a'b'c'$, the three conics which pass through the intersection of Aa, Bb, Cc and touch any two sides of the triangle abc at the extremities of the third side.

It will intersect in the chord $a'bc$ the three conics which pass through the intersection of Aa, Bb', Cc' and touch any two sides of the triangle $ab'c'$ at the extremities of the third side.

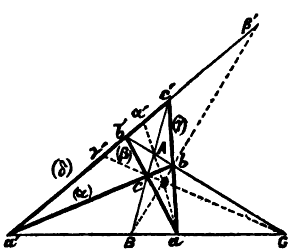
It will intersect in the chord $ab'c$ the three conics which pass through the intersection of Aa', Bb, Cc', and touch any two sides of the triangle $a'bc'$ at the extremities of the third side.

It will intersect in the chord abc' the three conics which pass through the intersection of Aa', Bb', Cc and touch any two sides of the triangle $a'b'c$ at the extremities of the third side.

☞ DEF.—*The critical conic of any quadrilateral is a circumscribed conic such that the tangent at any angular point forms a harmonic pencil with the sides and diagonal meeting at that point.*

It is obvious that if the quadrilateral be projected into a square, the critical conic will become the circumscribed circle.

Solution by the PROPOSER; *and* F. D. THOMSON, M.A.



Take bca', $b'ca$, abc', $a'b'c'$ as lines of reference for quadrilinear coordinates; and let the system of coordinates be that in which the quadrilinear coordinates of any point are connected together by the identity

$$\alpha+\beta+\gamma+\delta\equiv 0 \ldots\ldots\ldots\ldots (\text{i}).$$

Consider the equation of the second order

$$\beta\gamma+\alpha\delta+\gamma\alpha+\beta\delta+\alpha\beta+\gamma\delta = 0 \ldots (\text{ii});$$

this shall represent the conic required. For, in virtue of the identity (i), (ii) may be written

$$\beta\gamma+\alpha\delta+(\beta+\gamma)(\alpha+\delta)=0, \text{ or } \beta\gamma+\alpha\delta-(\beta+\gamma)^2=0;$$

which represents a conic having double contact in the line $(\beta+\gamma=0)$, which is known to be aa', with the conic whose equation is $\beta\gamma+\alpha\delta=0$, which is the critical conic of the quadrilateral $bb'cc'$.

But the equation (ii) may also be written

$$\gamma\alpha+\beta\delta-(\gamma+\alpha)^2=0$$

which represents a conic having double contact in the line $(\gamma+\alpha=0)$, or bb', with the conic whose equation is $\gamma\alpha+\beta\delta=0$, which is the critical conic of the quadrilateral $cc'aa'$.

Similarly by writing equation (ii) in the form

$$\alpha\beta+\gamma\delta-(\alpha+\beta)^2=0$$

we may establish the third property enunciated in the question.

Hence the equation (ii) represents the required conic.

Fourthly, the equation (ii) may be written

$$\beta\gamma+\gamma\alpha+\alpha\beta+\delta(\alpha+\beta+\gamma)=0, \text{ or } \beta\gamma+\gamma\alpha+\alpha\beta-\delta^2=0.$$

Hence the conic cuts the line $(\delta=0)$, or $a'b'c'$, in points determined by its intersection with $\beta\gamma+\gamma\alpha+\alpha\beta=0$, and therefore with

$$\beta\gamma-\alpha^2=0, \text{ or } \gamma\alpha-\beta^2=0, \text{ or } \alpha\beta-\gamma^2=0,$$

since when $\delta=0$, we have $\alpha+\beta+\gamma\equiv 0$.

But $\beta\gamma-\alpha^2=0$, $\gamma\alpha-\beta^2=0$, $\alpha\beta-\gamma^2=0$ represent conics touching two sides of the triangle abc at the extremities of the third, and passing through

the point determined by $\alpha=\beta=\gamma$, that is the point O determined by the intersection of the three straight lines

$$\beta-\gamma=0 \text{ (or A}a), \quad \gamma-\alpha=0 \text{ (or B}b), \quad \alpha-\beta=0 \text{ (or C}c).$$

In precisely the same way may the fifth and sixth properties be established. Hence the conic represented by the assumed equation passes through all the fourteen points mentioned in the Question.

[The results of Mr. Whitworth's paper on Quadrilinear Coordinates in the *Messenger of Mathematics*, Vol. I., p. 193, are assumed in the foregoing solution.]

II. *Solution by* T. A. Hirst, F.R.S.

The triangle Aaa' is self-conjugate relative to every conic circumscribed to $bb'cc'$; moreover, to the critical conic (C_1) of the system, which touches Bb at b, ABC is also a self-conjugate triangle; so that (C_1) divides aa' and BC harmonically, say in l and l'. Similarly the critical conic (C_2) circumscribed to $cc'aa'$ divides bb' and CA harmonically in m and m', and the critical conic (C_3) circumscribed to $aa'bb'$ divides cc' and AB harmonically in n and n'. The six points l, l', m, m', n, n' lie on one and the same conic (Σ) to which ABC, Aaa', Bbb', Ccc' are self-conjugate triangles. This known theorem is in our case manifest on observing, for instance, that n, n' must be the projections of m, m' from a, as well as from a'; so that a and a' are conjugate points relative to every conic through the four points m, m', n, n'; whence it follows that the conic of the pencil $(mm'nn')$ which passes through one of the points l, l', harmonic conjugates relative to aa', must pass through the other, &c. But BC being the polar of A relative to each of the conics (C_1), (Σ), which have BC for common chord, these conics must have double contact in l and l'. Similarly CA is the chord of contact, in m and m', of the conics (C_2) and (Σ), and AB is the chord of contact, in n and n', of the conics (C_3) and (Σ).

Further, if α', β', γ' be the intersections, with the line $a'b'c'$, of Aa, Bb, Cc, which latter are, of course, concurrent in O, it is manifest that $a'\alpha'$, $b'\beta'$, $c'\gamma'$, are three pairs of conjugate points relative to (Σ); they form, therefore, an involution whose foci are precisely the intersections of $a'b'c'$ with (Σ). Similar remarks apply to each of the three remaining sides of the quadrilateral $aa'bb'cc'$. The conic (Σ), in fact, is the fourteen-point conic of Prof. Cremona. (*Messenger of Mathematics*, Vol. III., p. 13.)

Consider now any one of the three conics which, passing through O, touch two of the sides of the triangle abc at the intersections of the latter with the third side; for instance, the conic (O_1) which touches ab, ac, at b and c. Since AO is divided harmonically by a and bc, polar of a relative to (O_1), the latter must not only pass through O but also through A. Accordingly the polars of a', b', c', relative to (O_1), are Aa', $c\beta'$, and $b\gamma'$, respectively; that is to say, $a'\alpha'$, $b'\beta'$, $c'\gamma'$ are three pairs of conjugate points relative to (O_1), as well as to (Σ); hence $a'b'c'$ must be the common chord of (O_1) and (Σ). The same may be proved with respect to the remaining two conics (O_2), (O_3), corresponding to the triangle abc.

The three last parts of the Question may be proved in a precisely similar manner, or inferred from the symmetry of the figure.

III. *Solution by* Professor Cayley.

1. The equations of the sides of the quadrilateral may be taken to be respectively $x=0$, $y=0$, $z=0$, $w=0$, where the implicit constants are so determined that we have *identically*

$$x+y+z+w=0;$$

this being so, the equations of the three diagonals are respectively

$$x+y=0, \text{ or } z+w=0, \text{ or } x+y-z-w=0 \text{ (three equivalent forms)}$$
$$x+z=0, \text{ or } y+w=0, \text{ or } x-y+z-w=0 \;(\;\text{,,}\qquad\text{,,}\qquad\text{,,}\;)$$
$$x+w=0, \text{ or } y+z=0, \text{ or } x-y-z+w=0 \;(\;\text{,,}\qquad\text{,,}\qquad\text{,,}\;)$$

and the equations of the critical conics are respectively

$$xy+zw=0,\quad xz+yw=0,\quad xw+yz=0.$$

Hence we see that the equation of the required conic is

$$\Delta = x^2+y^2+z^2+w^2-2yz-2zx-2xy-2xw-2yw-2zw=0.$$

In fact this equation may be written

$$\Delta = (x+y-z-w)^2 - 4(xy+zw) = 0,$$
$$\Delta = (x-y+z-w)^2 - 4(xz+yw) = 0,$$
$$\Delta = (x-y-z+w)^2 - 4(xw+yz) = 0,$$

equations which put in evidence the double contact with the three critical conics respectively. We have also, identically,

$$\Delta = (x+y+z+w)\,(x+y-3z-w) - 2w\,(x+y-z-w) + 4\,(z^2-xy),$$

and the equation $\Delta=0$ may therefore be written

$$\Delta = -2w\,(x+y-z-w) + 4\,(z^2-xy) = 0,$$

a form which shows that the conic $z^2-xy=0$ meets the line $w=0$ in the same two points in which it is met by the conic $\Delta=0$. And it hence appears by symmetry that the conics

$\Delta=0,\ x^2-yz=0,\ y^2-zx=0,\ z^2-xy=0$ meet the line $w=0$ in the same two points,

$\Delta=0,\ w^2-yz=0,\ y^2-zw=0,\ z^2-wy=0$ meet the line $x=0$ in the same two points,

$\Delta=0,\ w^2-xz=0,\ x^2-zw=0,\ z^2-xw=0$ meet the line $y=0$ in the same two points,

$\Delta=0,\ w^2-xy=0,\ x^2-yw=0,\ y^2-xw=0$ meet the line $z=0$ in the same two points,

which are the relations constituting the latter part of the proposed theorem.

2. The analogous theorems in space may be briefly referred to. Taking $x=0,\ y=0,\ z=0,\ w=0$ as the equations of the faces of a tetrahedron ABCD, then the implicit constants may be so determined that the coordinates of a given arbitrary point O shall be (1, 1, 1, 1). We may by lines drawn from the vertices of the tetrahedron project the point O on the faces, so as to obtain a point in each of the four faces; and then in each face, by lines drawn from the vertices of the face, project the point in that face upon the edges of the face; the two points thus obtained on each edge of the tetrahedron, are (it is easy to see) one and the same point; that is, we have on each edge of the tetrahedron a point; and there exists a quadric surface.

$$\Delta = x^2+y^2+z^2+w^2-2yz-2zx-2xy-2xw-2yw-2zw=0$$

touching the edges of the tetrahedron in these six points respectively.

The surface in question has plane contact with

the hyperboloid $xy+zw=0$ along the intersection with $x+y-z-w=0$,
,, ,, $xz+yw=0$,, ,, ,, $x-y+z-w=0$,
,, ,, $xw+yz=0$,, ,, ,, $x-y-z+w=0$,

and moreover the surfaces

$\Delta=0,\ x^2-yz=0,\ y^2-zx=0,\ z^2-xy=0$ meet the line $w=0,\ x+y+z+w=0$ in the same two points;

$\Delta=0,\ w^2-yz=0,\ y^2-zw=0,\ z^2-yw=0$ meet the line $x=0,\ x+y+z+w=0$ in the same two points;

$\Delta=0,\ w^2-xz=0,\ x^2-zw=0,\ z^2-xw=0$ meet the line $y=0,\ x+y+z+w=0$ in the same two points;

$\Delta=0,\ w^2-xy=0,\ x^2-yw=0,\ y^2-xw=0$ meet the line $z=0,\ x+y+z+w=0$ in the same two points.

With respect to the construction of the four planes,

$$x+y-z-w=0,\quad x-y+z-w=0,\quad x-y-z+w=0,\quad x+y+z+w=0,$$

it is to be observed that if through any edge of the tetrahedron, for instance the edge $x=0,\ y=0$, we draw the plan $x-y=0$ through the point O, then the harmonic of this in regard to the planes $x=0,\ y=0$ is the plane $x+y=0$, we have thus six planes, one through each edge of the tetrahedron, viz., these are $y+z=0,\ z+x=0,\ x+y=0,\ x+w=0,\ y+w=0,\ z+w=0$; the six planes being the faces of a hexahedron, which is such that the vertices of the tetrahedron are four of the eight vertices of the hexahedron: the pairs of opposite faces of the hexahedron meet in three lines lying in the plane $x+y+z+w=0$, and consequently forming a triangle such that through each side of the triangle there pass two opposite faces of the hexahedron; the planes $x+y-z-w=0,\ x-y+z-w=0,\ x-y-z+w=0$ are the harmonics of the plane $x+y+z+w=0$ in respect of the pairs of opposite faces of the hexahedron; viz., the plane $x+y-z-w=0$ is the harmonic of the plane $x+y+z+w=0$ in respect to the planes $x+y=0$, $z+w=0$; and the like for the other two planes $x-y+z-w=0$ and $x-y-z+w=0$ respectively.

1708. (Proposed by W. S. Burnside, B.A.)—

1. If the normals to a conic, drawn at the points A, B, C, D, meet in a point O; and if F be a focus of the conic, e the eccentricity, and $ke=b$; prove that FA . FB . FC . FD $= k^2 .$ FO^2.

2. If the normals to an ellipse at 1, 2, 3 meet in a point, and ω_{12} denote the angle which the chord (12) makes with an axis; prove that

$$\frac{\tan\omega_{12}}{\tan\omega_{33}}=\frac{\tan\omega_{23}}{\tan\omega_{11}}=\frac{\tan\omega_{31}}{\tan\omega_{22}}.$$

Solution by the Proposer; J. Dale; *and others.*

1. Let the coordinates of O be (α, β); then it is well known that A, B, C, D lie on the equilateral hyperbola $c^2xy+b^2\beta x-a^2\alpha y=0$ (Salmon's *Conics*, Art. 181, ex. 1); hence, eliminating y from this equation and $a^2y^2+b^2x^2-a^2b^2=0$, we have the following biquadratic in x for the intersections; viz.,

$$c^4x^4-2a^2c^2\alpha x^3+a^2(a^2\alpha^2+b^2\beta^2-c^4)x^2+2a^4c^2\alpha x-a^6\alpha^2=0.$$

Multiplying the roots of this equation by e, and substituting a for x, we have

$$c^4(a-ex_1)(a-ex_2)(a-ex_3)(a-ex_4) =$$

$$a^4c^4 - 2a^4c^3\alpha + a^2c^2(a^2\alpha^2 + b^2\beta^2 - c^4) + 2a^2c^5\alpha - c^4a^2\alpha^2 = a^2b^2c^2\left\{(\alpha-c)^2+\beta^2\right\},$$

therefore $$\text{FA}.\text{FB}.\text{FC}.\text{FD} = k^2.\text{FO}^2.$$

In the case of the parabola $y^2-4mx=0$, there are but three finite points A, B, C; and the relation is FA . FB . FC $= m$. FO².

2. Let α, β, γ be the eccentric angles of the points 1, 2, 3; then the equations of the normals are

$$\frac{ax}{\cos\alpha} - \frac{by}{\sin\alpha} = c^2,\ \frac{ax}{\cos\beta} - \frac{by}{\sin\beta} = c^2,\ \frac{ax}{\cos\gamma} - \frac{by}{\sin\gamma} = c^2;$$

hence, taking these in pairs and eliminating c^2, we have

$$\frac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\alpha} = \frac{\tan\frac{1}{2}(\gamma+\alpha)}{\tan\beta} = \frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\gamma},$$

which is equivalent to the relation in the Question. Also from any one of the last equations we obtain the condition

$$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0;$$

hence a geometrical interpretation of this relation.

In the case of the parabola, the condition for the normals meeting is $y_1+y_2+y_3=0$; and the relation is

$$\frac{y_2+y_3}{y_1} = \frac{y_3+y_1}{y_2} = \frac{y_1+y_2}{y_3} = -1.$$

On the Envelope in Question 1679.

(*Abridged from a paper by Steiner in the* 53rd *volume of Crelle's Journal.*)

Let ABC be a triangle; Aa, Bb, Cc the perpendiculars meeting in D; α, β, γ the middle points of the sides; m the centre of the nine-point circle (m_2), and r its radius; S the centre of the circumscribing circle (S_2). Then, m is the middle point of DS. The nine-point circle has three arcs outside the triangle; viz. αa, βb, γc. On these arcs respectively take three points u, v, w, so that arc $(\alpha u) = \frac{1}{3}$ arc (αa), arc $(\beta v) = \frac{1}{3}$ arc (βb), arc $(\gamma w) = \frac{1}{3}$ arc (γc). (1.) Then *the points uvw form an equilateral triangle.*

Let p be an arbitrary point on the circumference (S_2), and let G be the line joining the feet of perpendiculars from p on the sides of the triangle ABC. (2.) Then G *bisects* Dp *in* μ, *a point on the circle* (m_2). The circle (m_2) meets G in another point s; the points μ and s are called the *centre* and *vertex* of G respectively. (3.) Let p' be the point opposite to p on the circle (S_2), and G′ the line joining feet of perpendiculars from p'; then G′ *meets* G *at right angles at the point* s. G′ meets the circle (m_2) again in μ'; $\mu\mu'$ is a diameter of (m_2), and is parallel to pp'. Thus *for every tangent to the envelope* G_3 *there is one and only one tangent perpendicular to it; and the locus of their intersection s is the circle* (m_2). Two rectangular

tangents are called a *pair*. (4.) *Any line* G *is cut by any pair in two points equidistant from its centre* μ; whence *on every line* G *the point of contact* t *and the vertex* s *are equidistant from the centre* μ. (5.) *The chord of contact* tt' *of any pair* GG′ *is itself a line* G″ *of the system, and is of constant length* ($4r$). Hence every tangent G cuts the curve again in two points at a constant distance, the tangents at which are rectangular. (6.) Let t'' be the point of contact of G″; *the normals at* t, t', t'' *meet in a point, the locus of which is a circle* (m_2') *concentric with* (m_2), *and whose radius is* $3r$. (7.) *The curve* G_3 *touches the circle* (m_2) *at the points* u, v, w, *which are vertices of* G_3. (8.) Let the tangents at these points be U, V, W, then U′, V′, W′, *the tangents perpendicular to them respectively, are diameters of the circle* (m_2), *and are the cuspidal tangents of the curve* G_3. The centres of U′, V′, W′ are the points u', v', w', opposite to u, v, w, on the circle (m_2); so that the cusps u'', v'', w'', are at a distance $4r$ from the points u, v, w respectively. Hence the curve is situated symmetrically within the equilateral triangle $u''v''w''$, which is inscribed in the circle (m_2').

(9.) The whole length of the curve is $16r$, and its area is $2\pi r^2$.

(10.) Of two pairs GG′ and HH′, let G meet H, H′ in a', d', and let G′ meet H, H′ in b', c', respectively; then *the lines* $a'c'$, $b'd'$ *form another pair* JJ′. The points a', b', c', d' are such that each is the intersection of perpendiculars of the triangle formed by the other three. (11.) *For all such quadrangles the sum of the squares of opposite sides is constant*, and $= 16r^2 = a'd'^2 + b'c'^2 = a'c'^2 + b'd'^2 = a'b'^2 + c'd'^2$. If we start from a point d within the curve, we can draw three real tangents through it, and the tangents perpendicular to these will determine the *real* points a, b, c. But if the point d is outside the curve, only one real tangent G can be drawn through it, on which a real point a can be determined by (4). The perpendicular tangent G′ contains the imaginary points b, c, and the other two pairs H, H′ and J, J′ are imaginary.

(12.) *The circles circumscribing the four triangles* bcd, cda, dab, abc, a, b, c, d being any four points determined as in (10), *are equal, and have their common radius equal to* $2r$.

(13.) *The centres* L, M, N, S *of these circles form a quadrangle equal and similar to* $abcd$, *but inverted, and their centre of similitude is* m. *Any quadrangle* LMNS *determines a curve* Γ *inscribed in the circle* (m_2') *which is in fact merely the curve* G_3 *turned round through two right angles.* When b, c are imaginary, M, N are imaginary, and S is outside the curve Γ.

(14.) *Through each quadrangle* $abcd$ *passes a pencil of equilateral hyperbolas; the asymptotes of any one of these form a pair*, the vertex s being the centre of the hyperbola, and each pair determines a pencil of equilateral hyperbolas having these for asymptotes; so that the series of hyperbolas is doubly infinite. (15.) *Any two hyperbolas of the series intersect in a quadrangle* $abcd$, *such that* (ab, cd), (ac, db), (ad, bc) *are pairs*. (16.) *Any two such quadrangles lie in the same equilateral hyperbola.* (17.) *If two hyperbolas of the system touch, the point of contact is the centre* μ *of the common tangent.* Hence, given two right angles GG′ and HH′ in a plane, if two equilateral hyperbolas, having these respectively for asymptotes, touch one another, the point of contact μ lies on a fixed circle through the vertices of the right angles, and bisecting the segments which they determine on each other's rays. (18.) The system may also be defined thus. Let p be a point on the circle (m_2) and q any fixed line. Through each point s of the circle draw P through p, and Q parallel to q; then *the bisectors* GG′ *of the angle* PQ *envelop a curve* G_3 *equal to that we have been considering.*

(19.) In the circle (m_2) let a series of chords be inscribed as follows. The first ss_1 is taken arbitrarily; then s_1s_2 is drawn perpendicular to the diameter through s; s_2s_3 perpendicular to the diameter through s_1, and so on. *All these chords will touch the same curve* G_3. (20.) If the *arc* ss_1 is commensurable with the circumference of the circle, the series will return upon itself so as to form a closed polygon. Let $ss_1 : 2\pi = n : m$. The series will not necessarily return to the point s, but to some one of the points s, s_1, s_2 .... according to the form of the number m. The points $s, s_1, s_2, \ldots$ are vertices of a regular m-gon, and the chords are sides of this polygon of different orders (or sides and diagonals; a side of the rth order is a diagonal cutting off $r-1$ vertices on one side of it, so that a polygon of $2m+1$ vertices has $m-1$ orders of sides). (21.) The chord-polygon has for vertices all the vertices of the regular m-gon, and is itself an m-gon when m is a power of 3; its sides are then equal to each other three and three, and are sides of the complete regular m-gon of all those orders which are not divisible by 3. We have thus a polygon inscribed in the circle and circumscribed to the curve.

The following method of description is due to Professor Schäfli, of Bern. Consider a quadrangle $abcd$, each point being the intersection of the perpendiculars of the triangle formed by the other three; and let a series of conics be described, passing through the point d, and inscribed in the triangle abc. Through d draw a diameter dd_1 to each of these conics. (22.) Then the tangent G to this conic at the point d_1 will envelop a curve G_3, the same as we have already considered. The curve may also be described by rolling motion; it is, in fact, a hypocycloid of three branches.

Steiner then extends all this by projecting the whole figure orthogonally; in this paragraph, the only thing of importance is a new method of description. (23.) The points u, v, w are the three trisection-points of the arc μs cut off by any line G. Let then $m\mu$, ms be two arbitrary semi-diameters of an ellipse m_2; and let them move in different directions so that the sector described by ms in any time is double the sector described by $m\mu$ in the same time. (24.) Then the chord $s\mu$ envelops a three-cusped quartic touching the ellipse at the vertices of a maximum inscribed triangle, and having the line at infinity for a double tangent.

NOTES. (1) may be proved by showing, as is easily done, that $a\alpha + b\beta + c\gamma = 0$, when we pay attention to the signs. (2) and (3) are proved in the solution of Quest. 1649. For (5) and (6) see Quest. 1716; also *Lady's and Gentleman's Diary* for 1861, pp. 70—72. In (20), if α is the arc ss_1, measured from s, then the arc ss_n will be $\frac{1}{3}\left\{1-(-2)^n\right\}\alpha$. (23) follows immediately from the definition of the curve as a hypocycloid, which would probably be the simplest starting-point for the proof of all the rest. The following simple proof that the curve *is* a hypocycloid is due to Professor CAYLEY.

The equation of a three-cusped quartic is $(lx)^{-\frac{1}{2}} + (my)^{-\frac{1}{2}} + (nz)^{-\frac{1}{2}} = 0$; but this is completely determined when we have given x, y, z, and the ratios $l : m : n$, that is, the three cusps and the cuspidal tangents.

Another method of description is as follows:—If a conic be inscribed in the triangle $u''v''w''$ so as to pass through the centre m, then the lines joining u'', v'', w'', respectively to the opposite points of contact, intersect on the curve G_3.

In the 54th volume of Crelle's *Journal*, Professor Schröter noticed that the three-cusped quartic of Steiner was also the envelope of the connector of corresponding points of two anharmonically corresponding systems, one on a

circle, the other on the line at infinity; and hence he was led to interesting generalisations. Professor Cremona, in the current (64th) volume of the same *Journal*, has demonstrated, geometrically, all the above properties, and added some others.

1681. (Proposed by W. A. WHITWORTH, M.A.)—If $c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$ be a convergent recurring algebraical series of the rth order of recurrence, whose first $2r$ terms are given, prove that its sum to infinity will be

$$\begin{vmatrix} 0, & s_1x^{r-1}, & s_2x^{r-2} & \ldots\ldots\ldots & s_r \\ c_0, & c_1, & c_2 & \ldots\ldots\ldots & c_r \\ c_1, & c_2, & c_3 & \ldots\ldots\ldots & c_{r+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{r-1}, & c_r, & c_{r+1} & \ldots\ldots\ldots & c_{2r-1} \end{vmatrix} \div \begin{vmatrix} x^r, & x^{r-1}, & x^{r-2} & \ldots\ldots\ldots & 1 \\ c_0, & c_1, & c_2 & \ldots\ldots\ldots & c_r \\ c_1, & c_2, & c_3 & \ldots\ldots\ldots & c_{r+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{r-1}, & c_r, & c_{r+1} & \ldots\ldots\ldots & c_{2r-1} \end{vmatrix}$$

where $s_1, s_2 \ldots . s_r$ denote the sum of the first $1, 2 \ldots . r$ terms respectively.

Solution by JAMES DALE; *and* ALPHA.

Let the scale of relation be $1 + p_1x + p_2x^2 + \ldots . + p_rx^r$; then we have

$$p_rc_0x^r + p_{r-1}c_1x^r + \ldots\ldots\ldots\ldots + p_1c_{r-1}x^r + c_rx^r = 0,$$

$$p_rc_1x^{r+1} + p_{r-1}c_2x^{r+1} + \ldots . + p_1c_rx^{r+1} + c_{r+1}x^{r+1} = 0,$$

and so on, to infinity; hence, by addition, we get

$$p_rx^r(s_\infty - s_0) + p_{r-1}x^{r-1}(s_\infty - s_1) \ldots + p_1x(s_\infty - s_{r-1}) + (s_\infty - s_r) = 0;$$

therefore $$s_\infty = \frac{s_0p_rx^r + s_1p_{r-1}x^{r-1} + \ldots + s_{r-1}p_1x + s_r}{p_rx^r + p_{r-1}x^{r-1} + \ldots + p_1x + 1}.$$

Now $p_1, p_2, p_3, \ldots . p_r$ are given by the linear equations

$$c_0p_r + c_1p_{r-1} + c_2p_{r-2} + \ldots\ldots\ldots\ldots\ldots + c_{r-1}p_1 + c_r = 0,$$

$$c_1p_r + c_2p_{r-1} + c_3p_{r-2} + \ldots\ldots\ldots\ldots\ldots + c_rp_1 + c_{r+1} = 0,$$

$$\ldots\ldots\ldots\ldots$$

$$c_{r-1}p_r + c_rp_{r-1} + c_{r+1}p_{r-2} + \ldots\ldots + c_{2r-2}p_1 + c_{2r-1} = 0;$$

from which we obtain

$$p_r = \begin{vmatrix} c_1 & c_2 & \ldots c_{r-1} & c_r \\ c_2 & c_3 & \ldots c_r & c_{r+1} \\ c_3 & c_4 & \ldots c_{r+1} & c_{r+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_r & c_{r+1} & \ldots c_{2r-2} & c_{2r-1} \end{vmatrix} \div \begin{vmatrix} c_0 & c_1 & \ldots c_{r-2} & c_{r-1} \\ c_1 & c_2 & \ldots c_{r-1} & c_r \\ c_2 & c_3 & \ldots c_r & c_{r+1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{r-1} & c_r & \ldots c_{2r-3} & c_{2r-2} \end{vmatrix}$$

$$p_{r-1} = - \begin{vmatrix} c_0 & c_2 & \cdots & c_{r-1} & c_r \\ c_1 & c_3 & \cdots & c_r & c_{r+1} \\ c_2 & c_4 & \cdots & c_{r+1} & c_{r+2} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{r-1} & c_{r+1} & \cdots & c_{2r-2} & c_{2r-1} \end{vmatrix} \div \begin{vmatrix} c_0 & c_1 & \cdots & c_{r-2} & c_{r-1} \\ c_1 & c_2 & \cdots & c_{r-1} & c_r \\ c_2 & c_3 & \cdots & c_r & c_{r+1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{r-1} & c_r & \cdots & c_{2r-3} & c_{2r-2} \end{vmatrix}$$

with similar values for p_{r-2}, p_{r-3}, &c.

Substituting these values in the expression for s_∞, it becomes

$$\frac{s_0 x^r \begin{vmatrix} c_1 & c_2 & \cdots c_r \\ c_2 & c_3 & \cdots c_{r+1} \\ \vdots & \vdots & \vdots \\ c_r & c_{r+1} & \cdots c_{2r-1} \end{vmatrix} - s_1 x^{r-1} \begin{vmatrix} c_0 & c_2 & \cdots c_r \\ c_1 & c_3 & \cdots c_{r+1} \\ \vdots & \vdots & \vdots \\ c_{r-1} & c_{r+1} & \cdots c_{2r-1} \end{vmatrix} + \text{\&c.} \pm s_r \begin{vmatrix} c_0 & c_1 & \cdots c_{r-1} \\ c_1 & c_2 & \cdots c_r \\ \vdots & \vdots & \vdots \\ c_{r-1} & c_r & \cdots c_{2r-2} \end{vmatrix}}{x^r \begin{vmatrix} c_1 & c_2 & \cdots c_r \\ c_2 & c_3 & \cdots c_{r+1} \\ \vdots & \vdots & \vdots \\ c_r & c_{r+1} & \cdots c_{2r-1} \end{vmatrix} - x^{r-1} \begin{vmatrix} c_0 & c_2 & \cdots c_r \\ c_1 & c_3 & \cdots c_{r+1} \\ \vdots & \vdots & \vdots \\ c_{r-1} & c_{r+1} & \cdots c_{2r-1} \end{vmatrix} + \text{\&c.} \pm \begin{vmatrix} c_0 & c_1 & \cdots c_{r-1} \\ c_1 & c_2 & \cdots c_r \\ \vdots & \vdots & \vdots \\ c_{r-1} & c_r & \cdots c_{2r-2} \end{vmatrix}}$$

$$\therefore s_\infty = \begin{vmatrix} s_0 & s_1 x^{r-1} & \cdots\cdots & s_r \\ c_0 & c_1 & \cdots\cdots & c_r \\ c_1 & c_2 & \cdots\cdots & c_{r+1} \\ \vdots & \vdots & & \vdots \\ c_{r-1} & c_r & \cdots\cdots & c_{2r-1} \end{vmatrix} + \begin{vmatrix} x^r & x^{r-1} & \cdots\cdots & 1 \\ c_0 & c_1 & \cdots\cdots & c_r \\ c_1 & c_2 & \cdots\cdots & c_{r+1} \\ \vdots & \vdots & & \vdots \\ c_{r-1} & c_r & \cdots\cdots & c_{2r-1} \end{vmatrix}.$$

This coincides with the given form, since $s_0 = 0$.

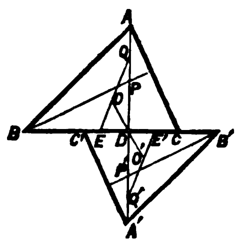
1659. (Proposed by J. Heyne.)—Given the base and vertical angle of a triangle, to construct it, so that the sum or difference of the perpendiculars from the ends of the base on the opposite sides may be equal to a given line, or so that the rectangle or sum or difference of the squares on these perpendiculars may be equal to a given square.

Solution by the Rev. R. Townsend, M.A.; E. Fitzgerald; *and others.*

This problem is identical with the following well-known one in Trigonometry; to divide an angle of given magnitude into two parts, such that the sum, difference, product, sum of squares, or difference of squares of the sines of these parts shall be given: of which several geometrical solutions are also well known. [See Townsend's *Modern Geometry*, Vol. I. Arts. 65—67.]

1660. (Proposed by J. GRIFFITHS, M.A.)—Let D be the foot of the perpendicular drawn from one of the vertices, A for instance, of any triangle ABC upon the opposite side BC. Produce AD to A′ so that AA′ = 2AD, and through A′ draw two lines A′B′, A′C′ parallel respectively to AB, AC. If these parallels intersect the side BC in B′ and C′, prove that the nine-point circles of the triangles ABC, A′B′C′ touch each other at the point D.

Solution by H. MURPHY; E. D. THOMSON, M.A.; J. DALE; *and others.*



Let P, P′ be the respective intersections of the perpendiculars of the triangles ABC, A′B′C′; E, E′ the middle points of the sides BC, B′C′; Q, Q′ the middle points of the segments AP, A′P′; and O, O′ the middle points of the lines QE, Q′E′; then O, O′ are the centres, and OD, O′D the radii, of the nine-point circles of the triangles ABC, A′B′C′. Now the triangles ODQ, O′DQ′ are, obviously, equal to each other in all respects; hence ODO′ is a *straight line*, and therefore the nine-point circles touch each other at the point D.

1661. (Proposed by R. WALSH.)—Show how to place 5 or 6 other figures on the right-hand side of 77777, so that the whole 10 or 11 figures may form a square number; and give a general rule for solving all such questions.

Solution by S. BILLS; E. FITZGERALD; P. W. FLOOD; *and others.*

The following method is perfectly *general* and strictly *direct.*

Take the case first of appending 6 figures. Annex 6 ciphers and extract the square root; then we have $\sqrt{(77777000000)} = 278885+$. Next add a unit to the last figure, annex 6 ciphers, and extract the square root; then $\sqrt{(77778000000)} = 278887+$. Now it is clear that if a square exists under the circumstances, its root must be greater than 278885, but not greater than 278887; that is, it must be either 278886, or 278887. Now $(278886)^2 = 77777400996$, and $(278887)^2 = 77777958769$; hence the figures to be added are either 400996, or 958769, consequently this case has *two* solutions.

Secondly, to annex 5 figures; we have

$$\sqrt{(7777700000)} = 88191+, \text{ and } \sqrt{(7777800000)} = 88191+.$$

Now from the first it appears that if a square exists under the circumstances its root must be greater than 88191, and from the second it is evident that its root cannot be greater than 88191; this case is therefore impossible.

As another example, taken at random, let there be given 9876 to annex, first, 4, and then 5 figures to make a square; we have

$$\sqrt{(98760000)} = 9937+, \text{ and } \sqrt{(98760000)} = 9938+;$$

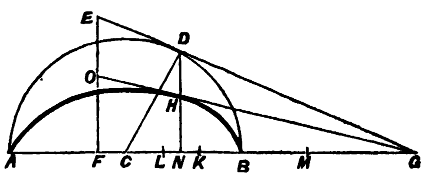
hence it follows that the root of the square will be 9938; and since $(9938)^2 = 98763844$, the figures to be added will be 3844.

Again, we have $\sqrt{(987600000)} = 31426+$, and $\sqrt{(987700000)} = 31427+$; the root of the square will therefore be 31427; and since $(31427)^2 = 987656329$, the figures to be annexed in this case will be 56329.

In each of the above cases there is one, and only one, solution. The same method is applicable if the enlarged number is required to be a cube, or any other power, instead of a square. To take an easy example; let it be required to annex 3 figures to 24, so that the extended number may be a cube. Here we have $\sqrt[3]{(24000)} = 28+$, and $\sqrt[3]{(25000)} = 29+$; hence the root of cube must be 29; and we have $(29)^3 = 24489$. The 3 figures to be annexed will therefore be 489. Any other question of this kind may be solved in like manner.

1665. (Proposed by E. CONNOLLY.)—To a given ellipse to draw a tangent, terminated by the major axis produced and a given ordinate to the same axis, such that the parts between the point of contact and the produced ordinate and axis may have to one another a given ratio.

Solutions (1) *by* E. FITZGERALD; (2) *by* J. TAYLOR.



1. Let AHB be the given ellipse, C its centre, FE the given ordinate, FC : CK the given ratio. To FC, FK, and CB^2 find a fourth proportional CM^2. Make CL = LK, and $LG^2 = CM^2 + CL^2$. Draw GDE touching the semicircle ADB in D; draw also DHN perpendicular to AB, and meeting the given ellipse in H; join GH and produce it to meet FE in O; then GHO, which is plainly a tangent to the ellipse, will be the tangent required.

For $\qquad CM^2 = LG^2 - CL^2 = CG \,.\, KG = CG^2 - CG \,.\, CK;$

therefore $\qquad FC : FK = CB^2 : CG^2 - CG \,.\, CK;$

therefore $\qquad FC \,.\, CG^2 = FK \,.\, CB^2 + FC \,.\, CG \,.\, CK;$

adding $CK \,.\, CG^2$, we have $FK \,.\, CG^2 = FK \,.\, CB^2 + FG \,.\, CG \,.\, CK;$

$\therefore\ FK \,.\, DG^2 = FG \,.\, CG \,.\, CK = EG \,.\, DG \,.\, CK$, $(\because FG \,.\, CG = EG \,.\, DG)$;

therefore $\qquad FK \,.\, DG = EG \,.\, CK$, or $EG : DG = FK : CK;$

hence, finally, $OH : HG = ED : DG = FC : CK =$ *the given ratio.*

2. *Otherwise*: Let $AC = CB = a$, $FC = -c$, $m : n =$ the given ratio, and $CN = h$; then $CG = a^2h^{-1}$; hence we have

$$m : n = FN : NG = h - c : a^2h^{-1} - h = h^2 - ch : a^2 - h^2;$$

therefore $$h = \frac{nc}{2(m+n)} \left\{ 1 \pm \sqrt{\left[4m(m+n)\left(\frac{a}{nc}\right)^2 + 1\right]} \right\}$$
which determines the position of the required tangent.

[NOTE.—The corresponding problem for the semicircle, to which that in the question may be at once reduced (since OH : HG = ED : DG, in the Figure), is a particular case of Quest. 1571, of which an elegant *analysis* by Mr. HOPPS is given on p. 21, of Vol. III. of the *Reprint*. See also Townsend's *Modern Geometry*, Vol. I. p. 48.—EDITOR.]

1663. (Proposed by T. DOBSON, B.A.)—If p_1, p_2, p_3 denote the perpendiculars, and r_1, r_2, r_3 the escribed radii of a plane triangle; prove that

$$\frac{p_2+p_3}{r_1} + \frac{p_3+p_1}{r_2} + \frac{p_1+p_2}{r_3} = 6.$$

Solutions by H. MURPHY; J. TAYLOR; J. DALE; P. W. FLOOD; E. FITZGERALD; D. M. ANDERSON; *the* PROPOSER; *and many others.*

It may be readily shown that the perpendicular on any side of a triangle is a harmonic mean between the radii of the circles escribed to the other two sides (see Mulcahy's *Modern Geometry*, p. 6; see also the Solution of Quest. 1698); hence we have

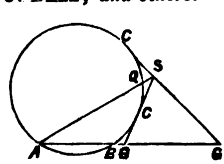
$$\frac{1}{r_2} + \frac{1}{r_3} = \frac{2}{p_1}; \;\therefore\; \frac{p_1}{r_2} + \frac{p_1}{r_3} = 2,\; \frac{p_2}{r_3} + \frac{p_2}{r_1} = 2,\; \frac{p_3}{r_1} + \frac{p_3}{r_2} = 2;$$
whence the theorem follows, by addition.

Otherwise: $\Delta = 2r_1s_1 = bp_2 = cp_3$, &c., whence we obtain

$$\frac{p_2+p_3}{r_1} + \frac{p_3+p_1}{r_2} + \frac{p_1+p_2}{r_3} = \left(\frac{2s_1}{b} + \frac{2s_1}{c}\right) + \left(\frac{2s_2}{c} + \frac{2s_2}{a}\right) + \left(\frac{2s_3}{a} + \frac{2s_3}{b}\right)$$
$$= \frac{2s_2 + 2s_3}{a} + \&c. = 2 + 2 + 2 = 6.$$

1654. (Proposed by T. T. WILKINSON, F.R.A.S.)—Draw, from a given point S, a straight line SG, meeting a given line AB at G, so that the rectangle AG . GB shall be equal to the square on the difference or sum of SG and a given line SC.

Solution by the PROPOSER; E. FITZGERALD; J. DALE; *and others.*



Join S with A, one of the given points in AB; in SA take a point Q such that AS . SQ = SC²; and draw a circle through A, B, Q; then a tangent SCG from S to this circle will obviously be the line required; for AS . SQ = the square on the given line = SC², and

$$AG \,.\, GB = GC^2 = (SG \pm SC)^2.$$

1635. (Proposed W. H. H. HUDSON, M.A.)—A pack of N cards is shuffled in any manner whatever, and then again in a precisely similar manner, and so on; show how to find after how many shufflings at most the cards will return to their original position.

If $N = 52$, the utmost number of shufflings is 180180.

Solution by the PROPOSER.

Consider the cards $A_1, A_2, A_3 \ldots A_n$ which, in n successive shufflings, occupy any particular position; then A_3 succeeds A_2 at the second shuffling by the same law by which A_2 succeeds A_1 at the first; hence it follows that A_3 previous to the second shuffling must have been in the place occupied by A_2 previous to the first. Thus the cards which successively occupy the place of A_2 are $A_2, A_3, \ldots A_n, A_1$; and so for all the original pieces of the cards $A_1, A_2 \ldots A_n$: these then form a cycle of n cards one or other of which is always in one or other of n positions in the pack, and which go through all their changes in n shufflings. Let the number N of the pack be divided into $n, n', n'' \ldots$ whose sum is N; then the greatest possible L. C. M. of $n, n', n'' \ldots$ is the utmost number of shufflings before the cards will be all brought back to their original places.

In the case of a pack of 52 cards, the greatest L. C. M. of numbers whose sum is 52 will be found by trial to be 180180.

1650. (Proposed by the Rev. J. BLISSARD)—Prove that

$$(1.)\ldots \cos (n+r)\,\theta \left(\frac{\sin n\theta}{\sin r\theta}\right) - \tfrac{1}{2}\cos (n+r)\, 2\theta \left(\frac{\sin n\theta}{\sin r\theta}\right)^2 + \tfrac{1}{3}\cos (n+r)\, 3\theta \left(\frac{\sin n\theta}{\sin r\theta}\right)^3 - \&c. = \log \left\{\frac{\sin (n+r)\,\theta}{\sin r\theta}\right\};$$

$$(2.)\ldots \sin (n+r)\,\theta \left(\frac{\sin n\theta}{\sin r\theta}\right) - \tfrac{1}{2}\sin (n+r)\, 2\theta \left(\frac{\sin n\theta}{\sin r\theta}\right)^2 + \&c. = n\theta.$$

Solution by J. BROWN; J. DALE; E. FITZGERALD; R. TUCKER, M.A.; *and others.*

Put $\dfrac{\sin n\theta}{\sin r\theta} = x$, $(n+r)\,\theta = \phi$, $S_1 =$ 1st series, $S_2 =$ 2nd series. Then

$$\log (1 + xe^{i\phi}) = x (\cos\phi + i \sin\phi) - \tfrac{1}{2} (\cos 2\phi + i \sin 2\phi) + \&c. \ldots\ldots\ldots (\alpha),$$

$$\log (1 + xe^{-i\phi}) = x (\cos\phi - i \sin\phi) - \tfrac{1}{2} (\cos 2\phi - i \sin 2\phi) + \&c. \ldots\ldots\ldots (\beta).$$

From $(\alpha) + (\beta)$ we get, after some easy reductions,

$$S_1 = \tfrac{1}{2}\log (1 + 2x\cos\phi + x^2) = \log \left\{\frac{\sin (n+r)\,\theta}{\sin r\theta}\right\}.$$

Moreover, putting $\frac{x \sin\phi}{1+x\cos\phi} = \lambda$, we have from $(\alpha)-(\beta)$,

$$2i\mathrm{S}_2 = \log\left(\frac{1+xe^{i\phi}}{1+xe^{-i\phi}}\right) = \log\left(\frac{1+i\lambda}{1-i\lambda}\right) = 2i(\lambda - \tfrac{1}{3}\lambda^3 + \tfrac{1}{5}\lambda^5 - \&c.) = 2i\tan^{-1}\lambda;$$

therefore $\qquad \mathrm{S}_2 = \tan^{-1}\lambda = \tan^{-1}(\tan n\theta) = n\theta.$

The series only hold good within the limits $(n+r)\,\theta = \pm\,\pi$.

1658. (Proposed by J. O'CALLAGHAN.)—A, B, C are three given points in the circumference of a given circle. It is required to draw from C a chord CD, such that if we divide it in a given ratio in E, and join AE, BE, the sum or difference of the squares on these lines may be given, a maximum, or a minimum.

Solution by the REV. R. TOWNSEND, M.A.; P. W. FLOOD; E. FITZGERALD; J. DALE; *the* PROPOSER; *and others.*

The locus of E being another circle touching the given circle at C, the problem is evidently the well-known one to find, on a given circle, a point such that the sum or difference of the squares on its distances from two given points A and B shall be given, a maximum, or a minimum.

1602. (Proposed by M. W. CROFTON, B.A.)—Prove that all conics which pass through both ends of the major and minor axes of an ellipse are cut orthogonally by a certain hyperbola, confocal with the ellipse.

Solution by the PROPOSER.

All conics which pass through both extremities of the two principal axes of an ellipse are represented by the equation

$$\mathrm{U} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + 2kxy - 1 = 0,$$

where k is a parameter; and these will be cut orthogonally, each in four points, by the hyperbola

$$\mathrm{V} = \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{a^2-b^2}{a^2+b^2} = 0$$

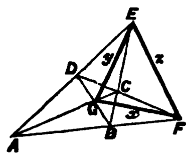
(which is confocal with the ellipse). For it is easy to show that, at the intersections, the condition

$$\frac{d\mathrm{U}}{dx}\cdot\frac{d\mathrm{V}}{dx} + \frac{d\mathrm{U}}{dy}\cdot\frac{d\mathrm{V}}{dy} = 0$$

will be satisfied. It may also be shown that the four tangents, at the intersections, to either curve, form a rectangle.

1555 (Proposed by W. A. WHITWORTH, M.A.)—The six straight lines joining the four points of contact with a conic section of common tangents to this conic and another, intersect, two and two, in the three points of intersection of the six common chords of the two conics, and form at these points harmonic pencils with the straight lines joining these points.

Solution by F. D. THOMSON, M.A.



Let E, F, G be the intersections of the three pairs of common chords of the two conics, and take EFG as the triangle of reference. Then, since each angular point of this triangle is the pole of the opposite side with respect to each conic, the equations to the two conics are of the forms

$$ax^2+by^2+cz^2=0 \ldots\ldots\ldots\ldots (1),$$
$$Ax^2+By^2+Cz^2=0 \ldots\ldots\ldots\ldots (2).$$

Let $lx+my+nz=0$ be the equation to a common tangent to (1) and (2); then if (x', y', z') be the point of contact with (1), we have

$$\frac{ax'}{l}=\frac{by'}{m}=\frac{cz'}{n}=\left(\frac{ax'^2+by'^2+cz'^2}{a^{-1}l^2+b^{-1}m^2+c^{-1}n^2}\right)^{\frac{1}{2}},$$

$$\therefore\ \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0 \ldots\ldots (3); \qquad \text{similarly } \frac{l^2}{A}+\frac{m^2}{B}+\frac{n^2}{C}=0 \ldots\ldots (4);$$

$$\therefore\ l^2 : m^2 : n^2 = \left(\frac{1}{bC}-\frac{1}{Bc}\right) : \left(\frac{1}{cA}-\frac{1}{Ca}\right) : \left(\frac{1}{aB}-\frac{1}{Ab}\right) \ldots\ldots\ldots (5).$$

Hence, taking l, m, n for the *positive* roots of $(bC)^{-1}-(Bc)^{-1}$ &c., the equations to the common tangents will be of the form

$$lx+my+nz=0,\ lx+my-nz=0,\ lx+my-nz=0,\ -lx+my-nz=0 \ldots (6).$$

Now if (x', y', z'), (x'', y'', z'') be the respective points of contact of the first and second of these common tangents, we have

$$\frac{ax'}{l}=\frac{by'}{m}=\frac{cz'}{n} \ldots (\alpha); \quad \frac{ax''}{l}=\frac{by''}{m}=\frac{-cz''}{n} \ldots\ldots (\beta),$$

and the equation to the line joining the points given by (α) and (β) is

$$x(y'z''-y''z')+y(z'x''-z''x')+z(x'y''-x''y')=0,$$

$$\text{or } -\frac{mn}{bc}x+\frac{nl}{ca}y=0, \quad \text{or } \frac{ax}{l}-\frac{by}{m}=0 \ldots\ldots (\gamma),$$

which is the equation to a line through the point G.

Similarly the equation to the line joining the points of contact of the third and fourth common tangent is found to be

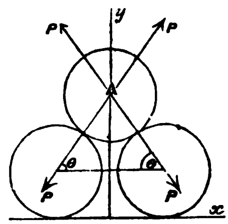
$$\frac{ax}{l}+\frac{by}{m}=0 \ldots\ldots\ldots\ldots\ldots (\delta).$$

Now (γ), (δ) form with $x=0$, $y=0$ a *harmonic* pencil. Similarly for the other lines joining the points of contact of common tangents.

1673. (Proposed by STEPHEN FENWICK, F.R.A.S.)—Two equal smooth spheres are held in contact on a smooth horizontal plane, and another smooth

equal sphere is placed upon them, so that the centres of the three spheres are in a vertical plane. The spheres being left to themselves, it is required to find the pressure on the upper one for any position of the spheres.

Solution by the PROPOSER.



Since the spheres and plane are smooth, and the centres of gravity of the spheres coincide with their geometrical centres, there is no rotatory motion; the whole motion in fact is one of translation. At a time t of the motion, let y be the distance from the horizontal plane of the centre A of the upper sphere, x the distance from a vertical line through A of the centre of either of the lower spheres, and P the reaction between the upper and either lower sphere. Also let the line of action of P, which is the normal to the upper and either of the lower spheres, be inclined to the horizon at an angle θ. Then m being the mass of each sphere, the respective equations for the motion of the upper and one of the lower spheres, will be

$$-m\frac{d^2y}{dt^2} = mg - 2\mathrm{P}\sin\theta \ldots\ldots (1); \quad m\frac{d^2x}{dt^2} = \mathrm{P}\cos\theta \ldots\ldots\ldots (2);$$

whence, eliminating P, $\quad 2\sin\theta\frac{d^2x}{dt^2} - \cos\theta\frac{d^2y}{dt^2} = g\cos\theta \ldots\ldots\ldots\ldots (3).$

But if r be the radius of each sphere, we have

$$x = 2r\cos\theta, \text{ and } y = r + 2r\sin\theta;$$

therefore $\quad \frac{dx}{dt} = -2r\sin\theta\frac{d\theta}{dt}, \quad \frac{d^2x}{dt^2} = -2r\cos\theta\frac{d\theta^2}{dt^2} - 2r\sin\theta\frac{d^2\theta}{dt^2};$

and $\quad \frac{dy}{dt} = \quad 2r\cos\theta\frac{d\theta}{dt}, \quad \frac{d^2y}{dt^2} = -2r\sin\theta\frac{d\theta^2}{dt^2} + 2r\cos\theta\frac{d^2\theta}{dt^2};$

hence (3) becomes $\sin\theta\cos\theta\frac{d\theta^2}{dt^2} + \sin^2\theta\frac{d^2\theta}{dt^2} = -\left(\frac{g}{2r}\cos\theta + \frac{d^2\theta}{dt^2}\right) \quad \ldots. (4).$

Integrating (4), we have $\sin^2\theta\frac{d\theta^2}{dt^2} = -\frac{g}{r}\sin\theta - \frac{d\theta^2}{dt^2} + \mathrm{C};$

hence, remembering that $\frac{d\theta}{dt} = 0$ when $\theta = 60^\circ$, we get

$$\frac{d\theta^2}{dt^2} = \frac{g}{2r}\cdot\frac{\sqrt{3} - 2\sin\theta}{1 + \sin^2\theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

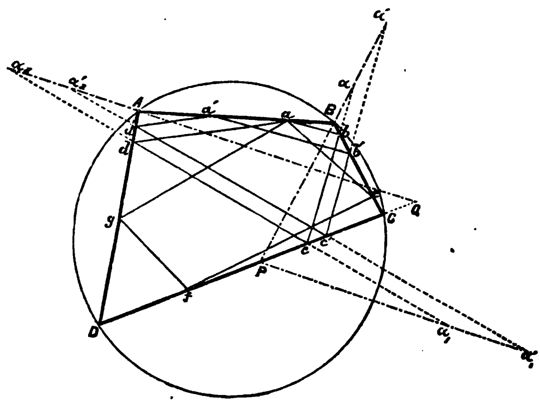
Again, substituting in (1), (2) the values of $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ given above, eliminating $\frac{d^2\theta}{dt^2}$ from the resulting equations, and solving for P, we get, by (5),

$$\mathrm{P} = mg\cdot\frac{\sin^3\theta + 3\sin\theta - \sqrt{3}}{(1 + \sin^2\theta)^2},$$

which gives the pressure on the upper sphere for any position of the spheres.

1705. (Proposed by H. MURPHY.)—If from the intersection of the diagonals of a quadrilateral inscribed in a circle perpendiculars be drawn on the sides, prove that the quadrilateral formed by joining the feet of these perpendiculars is, of all quadrilaterals inscribed in the given one, the one of least perimeter.

Solution by ARCHER STANLEY; J. DALE; E. FITZGERALD; *and others.*



ABCD being the given quadrilateral inscribed in a circle, the following two properties may be established :—

(1.) Of all quadrilaterals inscribed in the given one, and having one corner in common, that which has the least perimeter is determined by having its sides, at each corner, equally inclined to the sides of the given quadrilateral.

(2.) For different positions of the fixed corner, the inscribed quadrilaterals of minimum perimeter have their sides parallel, and their perimeters equal to one another.

Let a be the fixed corner referred to in (1), and let α be its optical image (or reflexion) relative to BC; that is to say, let Bα, which intersects CD in P, be equal to Ba, and equally inclined with the latter to BC. In the same manner let α_1 be the image of α relative to CD; so that Pα_1 is equal to Pα, and equally inclined with it to CD. Lastly, let α_2 be the image of a relative to DA, and let α_2A intersect CD in Q. Join α_2 and α_1 to cut DA, CD in d, c, and also c and α to cut BC in b. Then, on connecting a with b and d, a quadrilateral $abcd$ will be obtained whose sides are clearly equally inclined to those of ABCD. It is evident also that, a being fixed, this is the only quadrilateral possessing the property in question, and its perimeter is manifestly equal to the straight line $\alpha_1\alpha_2$. The perimeter of any other quadrilateral $aefg$ is equal to $\alpha e+ef+fg+g\alpha_2$, and is of course greater than $\alpha f+f\alpha_2$ or its equal $\alpha_1 f+f\alpha_2$; but the latter is greater than $\alpha_1\alpha_2$, the perimeter of $abcd$; so that the property (1) is rendered obvious.

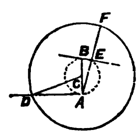
Let now a' be any other point of AB, and take its images α', α'_1, α'_2 as before, so as to construct, for a', the quadrilateral $a'b'c'd'$ of minimum peri-

meter. On observing that the angles CBP and ADQ, being supplements of the equal angles CBα and CBA, are equal to one another, and similarly that DAQ is equal to BCP, it will be evident that the triangles BPC and DQA are equiangular, so that BP and AQ are equally inclined to CD, and hence Q$\alpha_2\alpha'_2$ and P$\alpha_1\alpha'_1$ are parallel to each other; but, besides being parallel, $\alpha_1\alpha'_1$ and $\alpha_2\alpha'_2$ are also equal to one another, being images of the same segment $\alpha\alpha'$; hence $\alpha'_1\alpha'_2$ is equal and parallel to $\alpha_1\alpha_2$, and the property (2) follows as an immediate consequence.

The quadrilateral alluded to in the Question may be very readily proved to be one of those whose sides are equally inclined to those of the given quadrilateral ABCD. From the above properties it follows, therefore, that although no quadrilateral with a *less* perimeter can be inscribed, the number of inscribed quadrilaterals with *equally small* perimeters is infinite.

1701. (Proposed by A. RENSHAW.)—AD, AB are two lines at right angles to each other; AB is bisected in C, and CD is joined, D being a fixed point in AD; and with radius CD a circle is drawn round C as centre. Through B any line BE is drawn; also through A, AE is drawn perpendicular to BE and produced to meet the circle in F; prove that the rectangle AF . FE is constant.

Solution by D. M. ANDERSON; E. CONNOLLY; J. DALE; *the* PROPOSER; *and others.*



It is clear that if T be the tangent from F to a circle, radius = CA = CB, circumscribing the right-angled triangle AEB, we shall have

$$AF \cdot FE = T^2 = CF^2 - CA^2 = CD^2 - CA^2 = AD^2 = \text{a } constant.$$

1702. (Proposed by W. GODWARD.)—Prove that

$$\frac{ab}{(b-c)(c-a)}\tan^2\tfrac{1}{2}A\tan^2\tfrac{1}{2}B + \frac{bc}{(c-a)(a-b)}\tan^2\tfrac{1}{2}B\tan^2\tfrac{1}{2}C + \frac{ca}{(a-b)(b-c)}\tan^2\tfrac{1}{2}C\tan^2\tfrac{1}{2}A = -1.$$

Solution by E. CONNOLLY; D. M. ANDERSON; R. KNOWLES; J. DALE; E. FITZGERALD; *the* PROPOSER; *and others.*

Putting K for the left hand member in the Question, we have

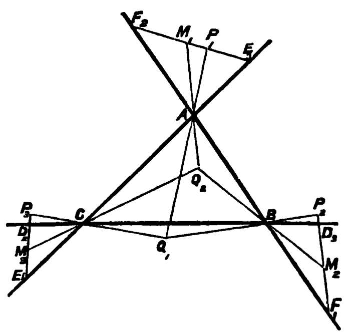
$$K = \frac{ab(a-b)(s-c)^2 + bc(b-c)(s-a)^2 + ca(c-a)(s-b)^2}{(a-b)(b-c)(c-a)s^2};$$

but $\quad ab(a-b) + bc(b-c) + ca(c-a) = -(a-b)(b-c)(c-a);$

therefore $\quad K = -1.$

1679. (Proposed by W. GODWARD.)—If (E_1, F_1), (F_2, D_2), (D_3, E_3) be the points of external contact of the escribed circles of a triangle ABC; and if AP_1, BP_2, CP_3 be drawn perpendicular to E_3F_2, F_1D_3, D_2E_1, respectively; it is required to prove that AP_1, BP_2, CP_3 will meet in a point, and to determine that point.

Solution by ALPHA; J. DALE; *and the* PROPOSER.



The coordinates of E_3, F_2 are, respectively (if $s_1 = s - a$, &c.),

$$(s \sin C,\ 0,\ -s_2 \sin A),$$
$$(s \sin B,\ -s_3 \sin A,\ 0);$$

hence the equation of E_3F_2 is

$$\alpha \sin^2 \tfrac{1}{2}A + \beta \cos^2 \tfrac{1}{2}B + \gamma \cos^2 \tfrac{1}{2}C = 0.$$

The equation of a line AP_1 through A perpendicular to E_3F_2 is

$$\frac{\beta}{\sin^2 \frac{1}{2}A - \sin^2 \frac{1}{2}B + \sin^2 \frac{1}{2}C} = \frac{\gamma}{\sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B - \sin^2 \frac{1}{2}C},$$

(with similar equations for BP_2 and CP_3); hence AP_1, BP_2, CP_3 meet in the point (Q_1 say) given by the equations

$$\frac{\alpha}{-\sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B + \sin^2 \frac{1}{2}C} = \frac{\beta}{\sin^2 \frac{1}{2}A - \sin^2 \frac{1}{2}B + \sin^2 \frac{1}{2}C} = \frac{\gamma}{\sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B - \sin^2 \frac{1}{2}C},$$

and since each of these three fractions is equal to each of the following

$$\frac{\Delta}{s_1 \sin^2 \frac{1}{2}A + s_2 \sin^2 \frac{1}{2}B + s_3 \sin^2 \frac{1}{2}C} = \frac{abc\, s\Delta}{2s\Delta^2} = \frac{abc}{2\Delta} = 2R,$$

the coordinates of the point Q_1 are

$$\alpha = 2R\left(-\sin^2 \tfrac{1}{2}A + \sin^2 \tfrac{1}{2}B + \sin^2 \tfrac{1}{2}C\right) = 2R\left(1 - 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C\right)$$
$$= 2R - a \sec \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C = 2R - r_1,$$

similarly $\qquad \beta = 2R - r_2, \text{ and } \gamma = 2R - r_3.$

It may be readily shown that this point Q_1, whose coordinates are $(2R - r_1, 2R - r_2, 2R - r_3)$, is the centre of the circle which passes through the centres O_1, O_2, O_3 of the escribed circles of the triangle ABC.

COR. 1.—From the equations of E_3F_2, F_1D_3, D_2E_1, we see that the intersections of E_3F_2 with BC, of F_1D_3 with CA, and of D_2E_1 with AB, lie on the straight line $\alpha \cos^2 \frac{1}{2}A + \beta \cos^2 \frac{1}{2}B + \gamma \cos^2 \frac{1}{2}C = 0$.

COR. 2.—The equation of E_3F_2 may be put in the form

$$(\alpha + \beta + \gamma) + (-\alpha \cos A + \beta \cos B + \gamma \cos C) = 0;$$

hence we see that, if R_1, R_2, R_3 are the feet of the perpendiculars from A, B, C on BC, CA, AB, respectively, the intersections of R_2R_3 with E_3F_2, of R_3R_1 with F_1D_3, and of R_1R_2 with D_2E_1, lie on the straight line $\alpha + \beta + \gamma = 0$, that is, on the straight line passing through the points in which the sides of the triangle are cut by the external bisectors of its angles.

Cor. 3.—If M_1, M_2, M_3 are the middle points of the lines E_3F_2, F_1D_3, D_2E_1, the equation of AM_1 is $s_2\beta - s_3\gamma = 0$, &c.; hence AM_1, BM_2, CM_3 meet in the point (Q_2 say) .

$$s_1\alpha = s_2\beta = s_3\gamma, \quad \text{or} \quad \frac{\alpha}{r_1} = \frac{\beta}{r_2} = \frac{\gamma}{r_3} = \frac{2\Delta}{ar_1 + br_2 + cr_3}.$$

1467. (Proposed by Hugh Godfray, M.A.)—n counters are marked with the numbers 1, 2, 3, ... n, respectively. Show that the number of ways in which three may be drawn, so that the greatest and least together may be double the mean, is

$$\tfrac{1}{4}n(n-2) + \tfrac{1}{8}\left\{1-(-1)^n\right\}, \quad \text{or} \quad \tfrac{1}{4}(n-1)^2 - \tfrac{1}{8}\left\{1+(-1)^n\right\}.$$

Solutions (1) *by* W. S. B. Woolhouse, F.R.A.S.; (2) *by the* Proposer *and* R. Tucker, M.A.

1. The numbers of the three counters drawn are required to be in arithmetical progression, and the interval or range between the two extremes must therefore be double the common difference; so that, according as the least number drawn is 1, 2, 3, &c., the number of ways will be respectively $\frac{1}{2}(n-1)$, $\frac{1}{2}(n-2)$, $\frac{1}{2}(n-3)$, &c., rejecting fractions; therefore the required number of ways is

$$\tfrac{1}{2}\left\{(n-1)+(n-1)+(n-3)\ \dots\ +1\right\} - \tfrac{1}{2}\,(\text{no. of odd numbers from 1 to } n-1)$$

$$= \tfrac{1}{2}\left\{\tfrac{1}{2}n(n-1) - \begin{matrix}\frac{1}{2}n \\ \frac{1}{2}(n-1)\end{matrix}\right\} = \begin{Bmatrix}\frac{1}{4}n(n-2) \\ \frac{1}{4}(n-1)^2\end{Bmatrix} \text{ according as } n \text{ is } \begin{Bmatrix}\text{even} \\ \text{odd}\end{Bmatrix},$$

and these values are both included in either of the proposed expressions.

2. *Otherwise*:—Let P_n denote the number of sets of three, satisfying the given condition, which can be formed with the n counters 1, 2, 3 .... n. It will be easily seen that the introduction of another counter marked $n+1$ will enable us to form $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ additional sets, according as n is even or odd.

Thus $P_{n+1} = P_n + \frac{1}{2}n$, or $= P_n + \frac{1}{2}(n-1)$, according as n is even or odd;

or, in all cases, $\qquad P_{n+1} = P_n + \frac{1}{2}n - \frac{1}{4} + \frac{1}{4}(-1)^n.$

This may be solved by the rules of finite differences, or simply by substituting successively 1, 2, 3, &c. for n, as follows:—

$$\begin{aligned}
P_2 &= P_1 + \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{4}(-1) \\
P_3 &= P_2 + \tfrac{2}{2} - \tfrac{1}{4} + \tfrac{1}{4}(-1)^2 \\
P_4 &= P_3 + \tfrac{3}{2} - \tfrac{1}{4} + \tfrac{1}{4}(-1)^3 \\
&\vdots \\
P_n &= P_{n-1} + \tfrac{1}{2}(n-1) - \tfrac{1}{4} + \tfrac{1}{4}(-1)^{n-1};
\end{aligned}$$

hence, adding, and remarking that $P_1 = 0$, we have

$$\begin{aligned}
P_n &= \tfrac{1}{2}\left\{1+2+\dots(n-1)\right\} - \tfrac{1}{4}(n-1) + \tfrac{1}{4}\left\{(-1)+(-1)^2+\dots(-1)^{n-1}\right\} \\
&= \tfrac{1}{4}n(n-1) - \tfrac{1}{4}(n-1) + \tfrac{1}{4}(-1)\frac{1-(-1)^{n-1}}{1-(-1)} \\
&= \tfrac{1}{4}(n-1)^2 - \tfrac{1}{8}\left\{1+(-1)^n\right\}.
\end{aligned}$$

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[TURN OVER

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